Spring 2019 Econ 164 Final Exam

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9 June, 2019

The exam has 5 questions for 100 points in total. You have 3 hours to finish the exam. The exam is open book.

ALL QUESTIONS ARE EQUALLY VALUABLE

DO NOT OPEN THE EXAM BOOK UNTIL THE EXAM BEGINS!

Name:

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1. Solow model and changing nature of capital .

Consider the Solow model with population and technology growth. The steady state capital per effective labor $\tilde{k}_{ss}$ is given by

$$\tilde{k}_{ss} = \left(\frac{s}{\delta + n + \tilde{x} + n\tilde{x}} \right)^{\frac{1}{1-\alpha}},$$

Assume $\alpha = 0.3$, $s = 0.25$, $\delta = 0.05$, $n = 0.01$, and $\tilde{x} = 0.01$. Answer the following questions. (20 points)

a. Find the numerical value of capital per effective labor in the steady state. (6 points)

$$\tilde{k} = (0.25/(0.05 + 0.01 + 0.01 + 0.01^2))^{1/0.7} \approx 6.15$$

b. Demonstrate that the steady state value $\tilde{k}_{ss}$ falls in response to an increase in the depreciation rate, in particular, to the value of 0.1. What happens to output per effective labor? (7 points)

$$\tilde{k} = (0.25/(0.1 + 0.01 + 0.01 + 0.01^2))^{1/0.7} \approx 2.85 < 6.15$$

Both capital and output per effective labor fall.

c. In practice, the capital stock of the economy includes many types of capital. One of the main technological advancements of the last two decades is the rise of digital economy, in which most of the capital stock has an intangible nature, like data creation and software: Think of Google—their main capital is your search history data and software that targets the ads using these data. It is established that software becomes obsolete very fast, so arguably the depreciation rate is now larger. Based on part (b), we might be tempted to conclude that capital and output per effective labor should fall according to the Solow model. Argue what other parameter(s) of the model might be affected by the rise of digital economy to invalidate this conclusion. (7 points)

The rise of digital economy might mean that the economy is becoming less labor-intensive, thus the share of labor in output falls, i.e. α goes up. Given that $\tilde{k} > 1$, it will increase capital per effective labor, like we saw in the midterm.

We can also hypothesize that digital economy is more innovative and thus technological growth $\tilde{x}$ will accelerate. Note, however, that while this will increase output per worker, it will have a negative effect on output per effective worker.

2.Solow model and investment efficiency

In class we have modeled technological progress as the term A_t that was affecting the efficiency of factors in the aggregate production function. This is also called Hicks-Neutral technological change. A prominent alternative is to model technological progress as an increase in investment efficiency.

Assume that the aggregate production function is $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$. Depreciation rate is δ. Constant fraction s of total output is saved and invested to produce new capital. Each unit of invested output produces q_t units of capital next period.

The term q_t corresponds to the investment efficiency—it tells us how much new capital we can get if we invest one unit of output today. In other words, technological progress is *embodied* in capital. In class we implicitly assumed that $q_t = 1$, i.e. output is converted to capital one-to-one. Here we relax this assumption and allow investment efficiency to change. Answer the following questions. (20 points)

a. Without making any further assumptions beyond that of q_t, write the law of motion for capital at $t + 1$ as the function of capital and labor at time t (in the context of the Solow model). (4 points)

$$K_{t+1} = (1 - \delta)K_t + sq_t A_t K_t^{\alpha} L_t^{1-\alpha}$$

b. Now assume that $L_t = L, A_t = A, q_t = q$ constant over time, i.e. there is no population growth and there is no technological progress of any kind. Rewrite law of motion using per-capita capital and solve for steady state capital per capita. Then use it to solve for output per capita and show that it will be equal to: (6 points)

$$y_{ss} = A^{\frac{1}{1-\alpha}} \left(\frac{qs}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

In per-capita variables,

$$k_{t+1} = (1-\delta)k_t + sqAk_t^{\alpha}$$

Setting $k_t = k_{t+1}$,we can solve for the steady state capital per capita:

$$k = (1-\delta)k + sqAk^{\alpha} \Rightarrow k = \left(\frac{sqA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Plugging this into per-capita production function $y_t = Ak_t^{\alpha}$ and factoring out A, we get the result.

c. How do the steady state capital and output per capita depend on investment efficiency q ? Argue using answer from previous question. (4 points)

It is increasing in q, which can be verified by taking derivative with respect to q or simply noticing that capital per capita is propotional to q taken to some positive power.

d. In reality, consumption and capital goods are very different goods. This change to the model captures that. Output can be consumed or invested, i.e. converted to capital goods. 1 unit of output delivers 1 unit of consumption if consumed, or q units of capital if invested. Imagine there is a competitive firm that converts consumption goods into capital goods. Let p_k and p_c denote the prices of consumption and capital goods, respectively, and the firm takes them as given. The firm solves the following problem:

$$\max_x \; p_k qx - p_c x$$

Argue that relative price of capital goods to consumption (e.g. relative price of computers to food) should be given by:

$$\frac{p_k}{p_c} = \frac{1}{q} \tag{1}$$

Below you can see scatter-plot of countries with relative price of capital goods (refering to $\frac{p_k}{p_c}$ in our equation) on vertical axis and output per capita on horizontal axis:

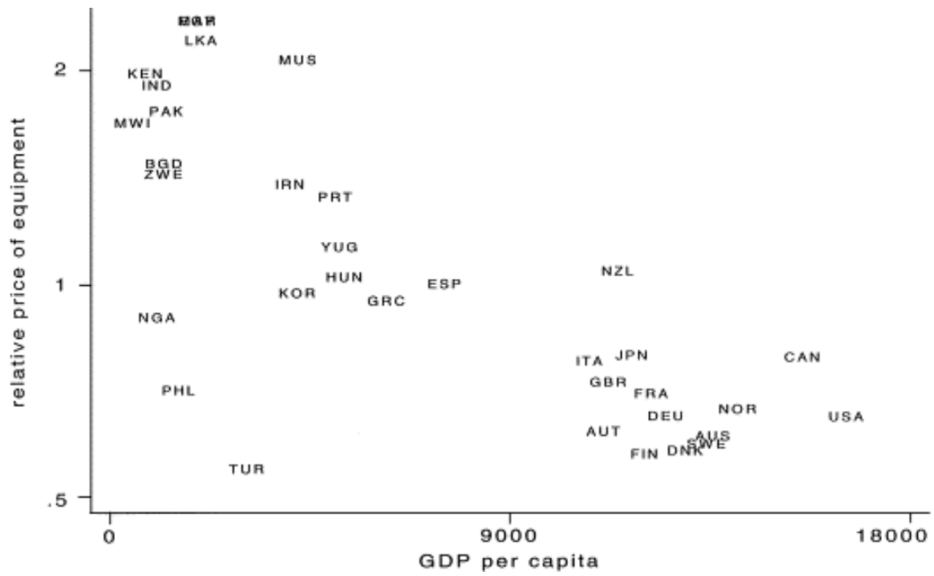


Figure 1: Relative price of capital and economic development

What pattern of the data can you see? What does it imply for q in light of the equation (1)? Given your answer to question c, argue whether differences in investment efficiency can explain differences in GDP per capita. (6 points)

The equation 1 follows immediately from taking first order conditions of the firm producing capital goods.

On the scatter-plot we see that richer countries tend to have lower relative price of capital goods. Given equation 1, this implies that richer countries have higher investment efficiency. As we determined in part (c), Solow model with investment efficiency implies that countries with higher investment efficiency should have higher GDP per capita, which is consistent with the data on lower relative price of capital goods in richer countries. So, differences in investment efficiency is a plausible explanation for cross-country differences.

3. Misallocation in Solow model

Governments can be a source of low TFP. Consider the following problem inside the Solow model. There are two sectors in an economy, $i = 1, 2$. They both produce output according to the following technologies:

$$Y_{1,t} = K_{1,t}^{\alpha}$$

and

$$Y_{2,t} = K_{2,t}^{\alpha}$$

The output of both sectors are equally valuable as consumption and investment. Thus, total output is the sum of output of the two sectors:

$$Y_t = Y_{1,t} + Y_{2,t}$$

The resource constraint of this economy is given by:

$$K_t = K_{1,t} + K_{2,t}.$$

where K_t is the time-varying aggregate capital stock. The setup of households follows the original Solow model we have in class . In particular, suppose the households own all of the capital and receive all the output as income, and save a fixed fraction s of their total income as investment. Each unit of invested income produces one unit of capital, and the existing capital stock depreciates at rate δ:

$$K_{t+1} = (1 - \delta)K_t + sY_t$$

For simplicity, we assume there is no technology and population growth, and the total population size is normalized to 1, so that per-capita variables are the same as aggregate variables: $k_t = K_t$, $c_t = C_t$ etc. Answer the following questions. (20 points)

a. Given K_t, find out the efficient allocation (the optimal $\{K_{1,t}, K_{2,t}\}$) of existing capital stock in a period t. Find out the efficient output in the same period. (4 points)

The efficient allocation maximizes total output given the resource constraint, i.e.

$$\max_{K_{1,t}} K_{1,t}^{\alpha} + (K_t - K_{1,t})^{\alpha}.$$

The first-order condition yields

$$\alpha K_{1,t}^{\alpha-1} = \alpha K_{2,t}^{\alpha-1} \rightarrow K_{1,t} = K_{2,t}.$$

Thus the efficient allocation is

$$K_{1,t}^{eff} = K_{2,t}^{eff} = \frac{1}{2} K_t.$$

b. Find out the steady-state level of capital per capita and consumption per capita under the efficient allocation described in part a. (4 points)

Under the efficient allocation the total output of the economy is

$$Y_t^{eff} = \left(\frac{1}{2} K_t\right)^{\alpha} \times 2 = 2^{1-\alpha} K_t^{\alpha}.$$

Thus the law of motion of capital stock per capita can be written as

$$k_{t+1} = (1-\delta) k_t + s \cdot 2^{1-\alpha} k_t^{\alpha}.$$

The steady-state capital per capita is given by:

$$k_{ss}^{eff} = 2 \left(\frac{s}{\delta}\right)^{1/(1-\alpha)}.$$

The steady-state consumption per capita is

$$c_{ss}^{eff} = 2 (1-s) \left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)}.$$

c. Now suppose the government taxes sector 1 for the output they produce and subsidizes sector 2. . Sector 1 gets to keep a fraction $1-\tau>0$ of its output and sector 2 gets to keep $1+\sigma>1$ of its output.. Solve the capital allocation under competitive equilibrium given the tax and subsidy in a period t. What is the total output of this economy? (4 points)

Denote the equilibrium rental rate of capital as r_t. Under competitive equilibrium, sector 1 maximizes

$$\max_{K_{1,t}} (1-\tau) K_{1,t}^{\alpha} - r_t K_{1,t}$$

which yields the first-order condition

$$K_{1,t} = \left[\frac{\alpha(1-\tau)}{r_t}\right]^{1/(1-\alpha)}.$$

Sector 2 maximizes

$$\max_{K_{2,t}} (1+\sigma) K_{2,t}^{\alpha} - r_t K_{2,t}$$

which yields the first-order condition

$$K_{2,t} = \left[\frac{\alpha(1+\sigma)}{r_t}\right]^{1/(1-\alpha)}$$

The equilibrium rental rate r_t is given by the capital market clearing condition:

$$K_t = \left[\frac{\alpha(1-\tau)}{r_t}\right]^{1/(1-\alpha)} + \left[\frac{\alpha(1+\sigma)}{r_t}\right]^{1/(1-\alpha)},$$

which implies

$$\left(\frac{1}{r_t}\right)^{1/(1-\alpha)} = \frac{K_t}{[\alpha(1-\tau)]^{1/(1-\alpha)} + [\alpha(1+\sigma)]^{1/(1-\alpha)}}.$$

Thus the equilibrium capital allocation is

$$K_{1,t} = \frac{(1-\tau)^{1/(1-\alpha)}}{(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}} K_t,$$

$$K_{2,t} = \frac{(1+\sigma)^{1/(1-\alpha)}}{(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}} K_t.$$

The total output is

$$Y_t = \frac{(1-\tau)^{\alpha/(1-\alpha)} + (1+\sigma)^{\alpha/(1-\alpha)}}{\left[(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}\right]^{\alpha}} K_t^{\alpha}.$$

d. Find out the steady-state level of capital per capita and consumption per capita under the allocation described in part c. (4 points).

We can follow the steps in part b except that replace the aggregate TFP $2^{1-\alpha}$ with

$$\frac{(1-\tau)^{\alpha/(1-\alpha)} + (1+\sigma)^{\alpha/(1-\alpha)}}{\left[(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}\right]^{\alpha}}.$$

The solutions are

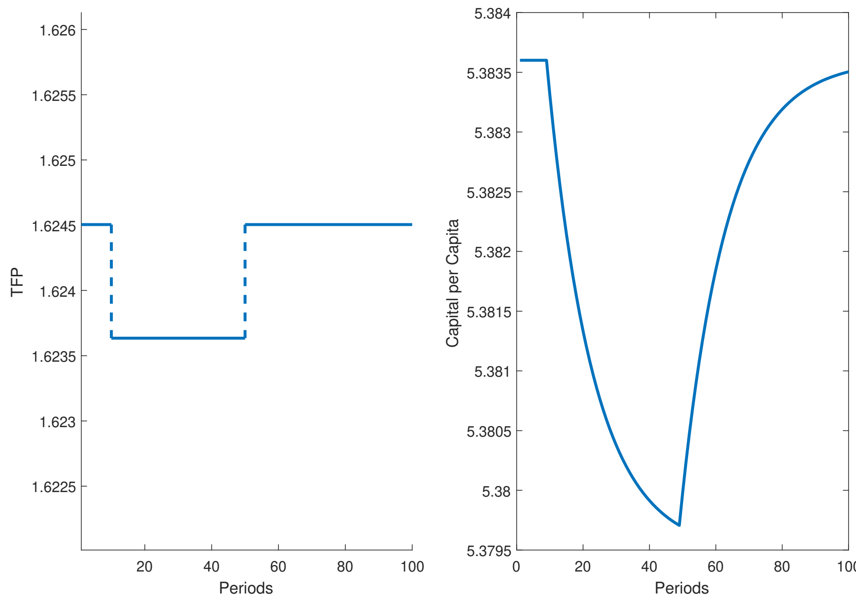
$$k_{ss} = \left\{ \frac{s}{\delta} \frac{(1-\tau)^{\alpha/(1-\alpha)} + (1+\sigma)^{\alpha/(1-\alpha)}}{\left[(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}\right]^{\alpha}} \right\}^{1/(1-\alpha)}$$

and

$$c_{ss} = (1-s)\left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)} \left\{ \frac{(1-\tau)^{\alpha/(1-\alpha)} + (1+\sigma)^{\alpha/(1-\alpha)}}{\left[(1-\tau)^{1/(1-\alpha)} + (1+\sigma)^{1/(1-\alpha)}\right]^{\alpha}} \right\}^{1/(1-\alpha)}.$$

e. Suppose the economy is initially in steady state from part b, and the government imposes taxes and subsidies as in part (d) at some moment t_1. Then at the moment $t_2 > t_1$ they lose the next election, and all taxes and subsidies are abolished. Taxes and subsidies are constant over time period $[t_1; t_2]$. Plot how you think the time path of (i) TFP and (ii) capital per capita will look like. (4 points)

The following figure plots the time paths of TFP and capital per capita, where we set $t_1 = 10$ and $t_2 = 50$. TFP temporarily drop during $[t_1, t_2]$, and capital per capita gradually decreases during this period, and gradually increases back to efficient steady state after t_2.



4. Borrowing constraints

An economy has a fixed capital stock K, which is distributed between two sectors of the economy, $i = 1, 2$. The total capital stock is equal to 100. The initial ownership of capital is split equally between two sectors. Denote the initial capital allocation as K_i for $i = 1, 2$. The two sectors produce output according to the following technologies:

$$Y_1 = A_1 K_1$$

and

$$Y_2 = A_2 K_2$$

where K_i is the actual capital used by sector i in production, and we have $A_2 = 2, A_1 = 1$, so sector 2 is more productive. Total output produced is the sum of output of the two sectors.

$$Y = Y_1 + Y_2$$

Answer the following questions. (20 points)

a. What is the efficient allocation of capital? What is the total output when the allocation is efficient?(4 points)

$$k_1 = 0,\ k_2 = 100,\ Y_{eff} = 2 * 100 = 200$$

b. Suppose there is borrowing constraint to each sector, such that

$$b_i \leq \lambda K_i$$

Argue that for some $\bar{\lambda}$, when $\lambda < \bar{\lambda}$, the competitive economy cannot achieve the efficient allocation, and find out $\bar{\lambda}$ given the initial allocation.(4 points)

In order for the competitive economy cannot achieve the efficient allocation, firm 2 has to borrow $b_2 = K_1 = 50$. This will not be possible if

$$50 > 50\lambda \iff \lambda < 1$$

So $\bar{\lambda} = 1$.

c. Suppose initially $\lambda = 0.5$. Show that $r = 1$ is the equilibrium interest rate. Compute total output and efficiency loss (i.e. by how many percent output is lower compared to the efficient level). Compute also profits of the firm in sector 2. (6 points)

If $r = 1$, firm 1 will be indifferent between borrowing and lending, up to the feasibility:

$$b_1 \in [0; 25]\,,\ s_1 \in [0; 50]$$

whereas the more productive firm 2 will be willing to borrow up to its borrowing constraint, and lend nothing:

$$b_2 = 25,\ s_2 = 0$$

Since firm 1 is indifferent, we can pick any amount of borrowing/lending in the feasible range. If we pick $b_1 = 0$ and $s_1 = 25$, markets for funds clear and thus we are in the equilibrium:

$$b_2 = 25 = s_1;\ b_1 = 0 = s_2$$

Total output is

$$Y_{dist} = 1 * (50 - 25) + 2 * (50 + 25) = 175$$

The efficiency loss is therefore:

$$Loss = (1 - \frac{Y_{dist}}{Y_{eff}}) * 100\% = 12.5\%$$

Finally, given that interest rate is equal to 1, profits of firm 2 are equal to

$$\Pi_2 = 2 * (50 + 25 - 0) - 1 * 25 = 125$$

d. Suppose that there is a possibility of a merger of firms in two sectors into one firm that has access to both technologies and owns all capital. Would this merger improve efficiency? Why or why not? (Hint: the merged entity will try to optimize allocation of capital between the two technologies available. Think which technology will they use and what will be the TFP of the new business) (2 points)

This new firm will only use superior technology of the firm 2, thus total output will be the efficient output. So it improves the efficiency.

e. Find equilibrium interest rate after the merger. (Hint: there is still market for borrowing and lending, but since now there is only one firm, in equilibrium it must be $b = 0$, $s = 0$. what should be the interest rate to make this new firm willing to borrow and lend nothing? Recall the TFP of the new firm from part (d)). What is the profits of the merged entity? If firms were able to merge at no cost and split equally the profits of the merged entity, would the firm in sector 2 be willing to do so? (4 points)

Merged firm will borrow or lend nothing only if it is indifferent between any borrowing or lending. To make the firm which has access to technology of firm 2 indifferent between borrowing or lending, it must be the case that $r = A_2 = 2$.

Of course, this has no effect on profits, as it borrows and lends nothing, and owns all capital:

$$\Pi_{merged} = 2 * 100 = 200$$

Firm 2 will receive $200/2 = 100$ of these profits, whereas in (c) it makes 125 of profits. So, firm 2 will not be willing to undertake this merger, even though it will improve efficiency from the social point of view.

5. Romer model

Consider the Romer model with finite-length patent. The economy consists of a final good sector and a group of capital sectors. The final good is produced by a group of price-taking firms in perfect competition, using the following production function:

$$Y_t = L^{1-\alpha}\left(\sum_{i=1}^{N_t} K_{i,t}^{\alpha}\right)$$

where L is the labor input that is constant over time, each $K_{i,t}$ is the capital produced by an monopolistic intermediate firm in capital sector i possessing the blueprints to produce that technology and effectively block others' access to it, N_t represents the number of intermediate sectors that exist in period t. The final good producer hires labor at wage w_t and rents capital from each monopolistic sector i at rental rate r_t^i. For simplicity, we assume the capital stock fully depreciates in each period.

An intermediate firm in a capital sector is able to produce one unit of capital using one unit of final good. We assume the patent length of an existing technology to produce a type of capital is T, which means the monopolistic intermediate firm that possesses the blueprint of this technology is able to reap the discounted sum of monopoly profits it generates in the future for T periods.

In each period, by investing Z_t units of final good into R&D, an intermediate firm can expand the number of varieties of capital stock by the amount:

$$\lambda\left(\frac{Z_t}{N_t}\right)^{1-\theta} N_t.$$

Answer the following questions. (20 points)

a. Write down the final good producer's first-order condition for labor input and each capital input. Write down the profit function and first-order condition of capital output for a monopolistic intermediate firm in capital sector i. Given these conditions, find out the expressions of the monopolistic intermediate firm's optimal choice of capital output, per period profit and the economy's final good output and equilibrium wage in terms of α, N_t and L. (10 points)

The final good producer maximizes its profit taking the wage rate and capital rental prices as given, i.e.

$$L^{1-\alpha}\left(\sum_{i=1}^{N_t} K_{i,t}^{\alpha}\right) - w_t L - \sum_{i=1}^{N_t} r_t^i K_{i,t}.$$

The first-order conditions with respect to L and $K_{i,t}$ are

$$(1-\alpha)\, L^{-\alpha}\left(\sum_{i=1}^{N_t} K_{i,t}^{\alpha}\right) - w_t = 0,$$

$$\alpha L^{1-\alpha} K_{i,t}^{\alpha-1} - r_t^i = 0.$$

The first-order condition for capital input implies that for each capital sector i, the final good firm is willing to pay:

$$r_t^i\left(K_{i,t}\right) = \alpha L^{1-\alpha} K_{i,t}^{\alpha-1}.$$

The profit function of the monopolistic intermediate firm in sector i is

$$\Pi_{i,t} = r_t^i\left(K_{i,t}\right) K_{i,t} - K_{i,t} = \alpha L^{1-\alpha} K_{i,t}^{\alpha} - K_{i,t}.$$

The first-order condition of capital output is

$$\alpha^2 L^{1-\alpha} K_{i,t}^{\alpha-1} = 1 \rightarrow K_{i,t} = \alpha^{\frac{2}{1-\alpha}} L,$$

and therefore per period profit is given by:

$$\Pi_{i,t} = \left(\alpha^{\frac{1+\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) L \equiv \mu L.$$

The total output of the economy is

$$Y_t = L^{1-\alpha} \left(\sum_{i=1}^{N_t} \alpha^{\frac{2\alpha}{1-\alpha}} L^{\alpha} \right) = \alpha^{\frac{2\alpha}{1-\alpha}} L N_t,$$

and the equilibrium wage is

$$w_t = (1-\alpha)\, \alpha^{\frac{2\alpha}{1-\alpha}} N_t.$$

b. Find out the expression of the value of a patent. (5 points)

The value of a patent is equal to the aggregate discounted profit earned by the patent holder, i.e. one monopolistic intermediate firm in capital good sector. Since the per period profit is a constant and the patent lasts for T periods, the patent value is given by the following equation:

$$V = \sum_{t=1}^{T} \beta^t \mu L = \beta \frac{1-\beta^T}{1-\beta} \mu L.$$

c. Write down the profit function of an intermediate firm doing R&D and the first-order condition of R&D input. Given this condition, find out the optimal choice of R&D input and the equilibrium growth rate of the economy. (5 points)

The profit function of doing R&D is

$$\Pi^{R\&D} = \lambda \left(\frac{Z_t}{N_t} \right)^{1-\theta} N_t \cdot V - Z_t.$$

The first-order condition of R&D input Z_t is

$$(1-\theta)\, \lambda \left(\frac{Z_t}{N_t} \right)^{-\theta} V = 1 \rightarrow \frac{Z_t}{N_t} = \left[(1-\theta)\, \lambda V \right]^{1/\theta} = \left[(1-\theta)\, \lambda \beta \frac{1-\beta^T}{1-\beta} \mu L \right]^{1/\theta}.$$

Therefore, the optimal R&D input is

$$Z_t = \left[(1-\theta)\, \lambda\beta \frac{1-\beta^T}{1-\beta} \mu L \right]^{1/\theta} N_t.$$

The equilibrium growth rate satisfies

$$1 + g = \frac{Y_{t+1}}{Y_t} = \frac{N_{t+1}}{N_t} = \frac{N_t + \lambda \left(\frac{Z_t}{N_t} \right)^{1-\theta} N_t}{N_t} = 1 + \lambda \left(\frac{Z_t}{N_t} \right)^{1-\theta},$$

which implies

$$g = \lambda \left[(1-\theta)\, \lambda\beta \frac{1-\beta^T}{1-\beta} \mu L \right]^{(1-\theta)/\theta}.$$