1 CES Operators

1.1 The Aggregator

- The CES operator, otherwise called Armington aggregator is defined:

$$V(W) = \left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta} \right)^{\frac{1}{1-\theta}}$$

subject to:

$$< p_i x_i >= W.$$

- θ is the inverse elasticity of substitution.
- Furthermore, $\sum \alpha_i = 1$, without loss of generality.
- We now find the solution and back out the wealth and substitution effects.
- First observation.
- Any solution to $V(W)$ is linear in W. The reason is that:

$$x_i = w_i W$$

for some weight. By change of variables:

$$\begin{aligned} V(W) &= \left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta} \right)^{\frac{1}{1-\theta}} \\ &= W \left(\sum_{i=1}^{N} \alpha_i^{\theta} w_i^{1-\theta} \right)^{\frac{1}{1-\theta}}. \end{aligned}$$

- We have the following FOC:

$$x_i : \frac{1}{1-\theta} V(W)^{\theta} (1-\theta) \frac{\alpha_i^{\theta}}{x_i^{\theta}} = \lambda p_i$$

Let's multiply by x_i to obtain:

$$\frac{V(W)^{\theta}}{\lambda} \frac{\alpha_i^{\theta}}{x_i^{\theta}} x_i = x_i p_i.$$

Summing up on all i, we obtain:

$$\sum_i \frac{V(W)^{\theta}}{\lambda} \frac{\alpha_i^{\theta}}{x_i^{\theta}} x_i = \sum_i x_i p_i = W.$$

Thus, we have that:

$$\begin{aligned}
\frac{V\left(W\right)^{\theta}}{W}\sum_{i}\frac{\alpha_{i}^{\theta}}{x_{i}^{\theta}}x_{i} &= \lambda\rightarrow \\
\lambda &= \frac{V\left(W\right)^{\theta}}{W}V\left(W\right)^{1-\theta}=\frac{V\left(W\right)}{W}.
\end{aligned}$$

Thus,

$$\begin{aligned}
W\frac{\alpha_{i}^{\theta}x_{i}^{1-\theta}}{\sum_{j}\frac{\alpha_{j}^{\theta}}{x_{j}^{\theta}}x_{j}} &= x_{i}p_{i} \\
x_{i}^{\theta} &= \frac{W}{p_{i}}\frac{\alpha_{i}^{\theta}}{\sum_{j}\frac{\alpha_{j}^{\theta}}{x_{j}^{\theta}}x_{j}} \\
x_{i} &= \alpha_{i}\left(\frac{W}{p_{i}}\frac{1}{\sum_{j}\frac{\alpha_{j}^{\theta}}{x_{j}^{\theta}}x_{j}}\right)^{\frac{1}{\theta}} \\
&= \alpha_{i}\left(\frac{W}{p_{i}}\frac{1}{V\left(W\right)^{1-\theta}}\right)^{\frac{1}{\theta}} \\
x_{i} &= \alpha_{i}\left(\frac{W}{p_{i}}\right)^{\frac{1}{\theta}}V\left(W\right)^{1-\frac{1}{\theta}}.
\end{aligned}$$

Back in objective:

$$x_{i}^{1-\theta}=\alpha_{i}^{1-\theta}\left(\frac{W}{p_{i}}\right)^{\frac{1-\theta}{\theta}}\left[V\left(W\right)^{1-\frac{1}{\theta}}\right]^{1-\theta}$$

Multiplying by: α_i we obtain:

$$\alpha_{i}^{\theta}x_{i}^{1-\theta}=\alpha_{i}\left(\frac{W}{p_{i}}\right)^{\frac{1-\theta}{\theta}}\left[V\left(W\right)^{1-\frac{1}{\theta}}\right]^{1-\theta}.$$

Thus,

$$\sum_{i}\alpha_{i}^{\theta}x_{i}^{1-\theta}=\sum_{i}\alpha_{i}^{\theta}x_{i}^{1-\theta}=\left[V\left(W\right)^{1-\frac{1}{\theta}}\right]^{1-\theta}\sum_{i}\alpha_{i}\left(\frac{W}{p_{i}}\right)^{\frac{1-\theta}{\theta}}.$$

Thus,

$$\begin{aligned}
V\left(W\right) &= \left[V\left(W\right)^{1-\frac{1}{\theta}}\right]\left(\sum_{i}\alpha_{i}\left(\frac{W}{p_{i}}\right)^{\frac{1-\theta}{\theta}}\right)^{\frac{1}{1-\theta}} \\
V\left(W\right) &= \left(\sum_{i}\alpha_{i}\left(\frac{1}{p_{i}}\right)^{\frac{1-\theta}{\theta}}\right)^{\frac{\theta}{1-\theta}}W.
\end{aligned}$$

The term

$$p^* = \left(\sum_i \alpha_i \left(p_i\right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$$

is called the ideal price index.

- Also, call

$$\begin{aligned} C^* &= V\left(W\right) \\ P^* C^* &= W. \end{aligned}$$

- Thus,

$$V\left(W\right) = \left(p^*\right)^{-1} W.$$

What is then x_i? From FOC:

$$\begin{aligned} x_i &= \alpha_i \left(\frac{W}{p_i}\right)^{\frac{1}{\theta}} V\left(W\right)^{1-\frac{1}{\theta}} \\ &= \frac{\alpha_i}{p_i^{1/\theta}} \left(p^*\right)^{\frac{1-\theta}{\theta}} W \\ &= \frac{\alpha_i}{p_i^{1/\theta}} \left(p^*\right)^{\frac{1}{\theta}} W \left(p^*\right)^{-1}. \end{aligned}$$

Thus we obtain the following relationship:

$$\frac{x_i}{C^*} = \frac{\alpha_i}{p_i^{1/\theta}} \left(p^*\right)^{\frac{1}{\theta}}.$$

Let $\mathrm{c}_i = x_i$ to obtain:

$$\frac{c_i}{C^*} = \alpha_i \left(\frac{p^*}{p_i}\right)^{\frac{1}{\theta}}.$$

1.2 Elasticitities

- Since there are N commodities. We are interested in the direct elasticities and the cross-elasticities across goods:

$$\in_i^i = \frac{\partial c_i}{\partial p_i} \frac{p_i}{c_i} \text{ and } \in_j^i = \frac{\partial c_i}{\partial p_j} \frac{p_j}{c_i}.$$

- Also, there are the Hicksian elasticities obtained when the agent is compensated with more wealth:

$$\bar{\in}_i^i = \frac{\partial c_i}{\partial p_i} \frac{p_i}{c_i} \text{ and } \bar{\in}_j^i = \frac{\partial c_i}{\partial p_j} \frac{p_j}{c_i}$$

where the formula for c changes to account for for the change in p.

- Note that neither concept is write nor wrong, they are just useful ways to view the world.
- The Hicksian compensated demand is given by:

$$c_i = \alpha_i \left(\frac{p^*}{p_i}\right)^{\frac{1}{\theta}} C^*$$

while holding C^* fixed. The idea is to keep "utility" constant. That is:

$$p^* C^* = W$$

requirese that:

$$\frac{\partial p^*}{\partial p_i} C^* = \frac{\partial W}{\partial p_i}.$$

Thus,

$$\epsilon_i^W = \frac{\partial W}{\partial p_i} \frac{p_i}{W} = \frac{\partial p^*}{\partial p_i} \frac{p_i}{1} \frac{C^*}{W} = \epsilon_i^*.$$

In other words, wealth goes up in the same proportion as prices.

- Thus, the Hicksian elasticity is obtained via:

$$\begin{aligned} \frac{\partial c_i}{\partial p_i} &= \frac{1}{\theta} C^* \alpha_i \left(\frac{p^*}{p_i}\right)^{\frac{1}{\theta}} \left[\frac{\partial p^*}{\partial p_i} \frac{1}{p^*} - p_i^{-1}\right] \rightarrow \\ \frac{\partial c_i}{\partial p_i} \frac{p_i}{c_i} &= \frac{1}{\theta} \left[\frac{\partial p^*}{\partial p_i} \frac{p_i}{p^*} - 1\right] \rightarrow \\ \bar{\epsilon}_i^i &= \frac{1}{\theta} \left[\epsilon_i^* - 1\right]. \end{aligned}$$

- In the case of the the cross-Hicksian demand:

$$\begin{aligned} \frac{\partial c_i}{\partial p_j} &= \frac{1}{\theta} C^* \alpha_j \left(\frac{p^*}{p_j}\right)^{\frac{1}{\theta}} \left[\frac{\partial p^*}{\partial p_j} \frac{1}{p^*}\right] \rightarrow \\ \bar{\epsilon}_j^i &= \frac{1}{\theta} \left[\epsilon_j^*\right]. \end{aligned}$$

- The inverse-demand (Marshallian demand) for good i take into account the effect on C^*. Thus, it's more convenient to work with the formula that uses wealth:

$$x_i = \frac{\alpha_i}{p_i^{1/\theta}} (p^*)^{\frac{1-\theta}{\theta}} W$$

$$\begin{aligned} \frac{\partial c_i}{\partial p_i} &= -\frac{1}{\theta} \frac{\alpha_i}{p_i^{1/\theta}} (p^*)^{\frac{1-\theta}{\theta}} W \frac{1}{p_i} + \frac{1-\theta}{\theta} \frac{\alpha_i}{p_i^{1/\theta}} (p^*)^{\frac{1-\theta}{\theta}} W \frac{1}{p^*} \\ &= -\frac{1}{\theta} c_i \frac{1}{p_i} + \frac{1-\theta}{\theta} c_i \frac{1}{p^*} \frac{\partial p^*}{\partial p_i} \frac{p_i}{p_i}. \end{aligned}$$

Hence, we obtain:

$$\epsilon_i^i = \frac{\partial c_i}{\partial p_i}\frac{p_i}{c_i} = \frac{1}{\theta}\underbrace{[\epsilon_i^* - 1]}_{\text{Substitution}} - \underbrace{\epsilon_i^*}_{\text{Wealth Effect}}.$$

- For the cross-elasticities we obtain:

$$\begin{aligned}\frac{\partial c_i}{\partial p_i} &= \frac{1-\theta}{\theta} c_i \frac{1}{p^*}\frac{\partial p^*}{\partial p_i}\frac{p_j}{p_j} \rightarrow \\ \epsilon_j^i &= \frac{1}{\theta}\underbrace{\epsilon_j^*}_{\text{Substitution}} - \underbrace{\epsilon_j^*}_{\text{Wealth Effect}}.\end{aligned}$$

- Excercis: Show that $\epsilon_j^* > 0$.
- Question: what happens when $1/\theta < 1$. Wealth effect dominates and $\epsilon_j^i < 0$.
- In turn, if $1/\theta > 1$, we have that $\epsilon_j^i > 0$.
- In turn, if $1/\theta = 1$, we have that $\epsilon_j^i = 0$.
- What if $1/\theta < 0$? We run into problems since operator is convex. Still, we can use the to show which (unique) good would be bought and determine Utility as function of the change in p_i.

1.3 Deriving Limits

- In mathematics,

$$\left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right)^{\frac{1}{1-\theta}}$$

is called the generalized mean.

- When $\theta = -1$, we have the Euclidean Norm.
- Note that when $\theta = 0$, we have a weigthed average.
- $\theta = 1/2$ we have:

$$\left(\sum_{i=1}^{N} \alpha_i^{1/2} x_i^{1/2}\right)^2.$$

- When $\theta = 2$, we obtain a geometrice mean:

$$\left(\sum_{i=1}^{N} \alpha_i^{2} x_i^{-1}\right)^{-1}.$$

We now derive some special limits:

1.4 Log-Utility Case

$$\begin{aligned}
& \lim_{\theta\to 1}\left(\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\right)^{\frac{1}{1-\theta}} \\
= & \lim_{\theta\to 1}\exp\left[\frac{1}{1-\theta}\log\left(\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\right)\right] \\
= & \exp\left[\lim_{\theta\to 1}\frac{1}{1-\theta}\log\left(\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\right)\right].
\end{aligned}$$

We can pass limits since exp is a continuous function (See Rudin).

Since

$$\lim_{\theta\to 1}\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta} = 1.$$

The term inside the exp is of the form 0/0. We can apply L'Hospital. Thus:

$$\begin{aligned}
& \lim_{\theta\to 1}\frac{1}{1-\theta}\log\left(\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\right) \\
= & \lim_{\theta\to 1}\frac{1}{\frac{1}{-1}}\frac{\sum_{i=1}^{N}\alpha_i^{\theta}\left(\log\alpha_i\right)x_i^{1-\theta}-\alpha_i^{\theta}\left(\log x_i\right)x_i^{1-\theta}}{\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}} \\
= & .\lim_{\theta\to 1}\frac{1}{1}\frac{-1\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\left[\left(\log\alpha_i\right)-\log x_i\right]}{\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}} \\
= & .\lim_{\theta\to 1}\frac{\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\left[\log\left(x_i/\alpha_i\right)\right]}{\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}} \\
= & \frac{\lim_{\theta\to 1}\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\left[\log\left(x_i/\alpha_i\right)\right]}{\lim_{\theta\to 1}\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}} \\
= & \sum_{i=1}^{N}\lim_{\theta\to 1}\alpha_i^{\theta}x_i^{1-\theta}\left[\log\left(x_i/\alpha_i\right)\right] \\
= & \log\left[\prod_{i=1}^{N}\left(\frac{x_i}{\alpha_i}\right)^{\alpha_i}\right]
\end{aligned}$$

I used the chain rule, and the fact that the derivative of x_i^{α} w.r.t. α is $x^{\alpha}\log x$. Since the limit was inside and exponential:

$$\lim_{\theta\to 1}\left(\sum_{i=1}^{N}\alpha_i^{\theta}x_i^{1-\theta}\right) = \prod_{i=1}^{N}x_i^{\alpha_i}.$$

The function becomes Cobb-Douglass with a scalar adjustment:

$$\left(\prod_{i=1}^{N} \alpha_i^{\alpha_i}\right)^{-1} \left[\prod_{i=1}^{N} x_i^{\alpha_i}\right].$$

1.5 Leontieff-Knightian Limit

$$\begin{aligned}
& \lim_{\theta\to\infty} \left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right)^{\frac{1}{1-\theta}} \\
= & \lim_{\theta\to\infty} \exp\left[\frac{1}{1-\theta} \log\left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right)\right] \\
= & \exp\left[\lim_{\theta\to\infty} \frac{1}{1-\theta} \log\left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right)\right].
\end{aligned}$$

This function is again of the form 0/0 (why?).

Then, applying L'Hospital:

$$\begin{aligned}
& \lim_{\theta\to\infty} \frac{1}{1-\theta} \log\left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right) \\
= & \left[\lim_{\theta\to\infty} \frac{1}{(1-\theta)^2} \left[\log\left(\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}\right)\right]^{-2}\right] \lim_{\theta\to\infty} \frac{\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta} \left[\log\left(x_i/\alpha_i\right)\right]}{\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}}.
\end{aligned}$$

First case: one $\alpha_i = 1$. Then, clearly, $U(x) = x_i$. So in the interesting case:

$$\begin{aligned}
\frac{\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta} \left[\log\left(x_i/\alpha_i\right)\right]}{\sum_{i=1}^{N} \alpha_i^{\theta} x_i^{1-\theta}} &= \sum_{i=1}^{N} \frac{\alpha_i^{\theta} x_i^{1-\theta}}{\sum_{j=1}^{N} \alpha_j^{\theta} x_j^{1-\theta}} \log\left(x_i/\alpha_i\right) \\
&= \sum_{i=1}^{N} \frac{1}{\sum_{j=1}^{N} \frac{\alpha_j^{\theta} x_j^{1-\theta}}{\alpha_i^{\theta} x_i^{1-\theta}}} \log\left(x_i/\alpha_i\right) \\
&= \sum_{i=1}^{N} \frac{1}{\sum_{j=1}^{N} \frac{x_j}{x_i} \left(\frac{x_i}{\alpha_i} / \frac{x_j}{\alpha_j}\right)^{\theta}} \log\left(x_i/\alpha_i\right).
\end{aligned}$$

Note that when we take the limit each side each summation, at least one term disappears when $\left(\frac{x_i}{\alpha_i} / \frac{x_j}{\alpha_j}\right) > 1$. So whenever, $\frac{x_i}{\alpha_i}$ is bigger than at least one element, the entire term disappars. The only survivor term is: $\log\left(x_i/\alpha_i\right)$ for x_i/α_i the min. Now if two terms are equal, of more than two (n) are equal. We end up with summations of the form:

$$n\frac{1}{n}\log\left(x_i/\alpha_i\right)$$

for x_i/α_i the $\min\{x_i/\alpha_i\}$.

1.6 Questions

- Derive the case where $\theta \to -\infty$.

1.7 Infinite Countable Case

1.8 Integrable Case

1.9 Recursive Properties of the Aggregator

1.10 Aggregation Properties

- Statement of Gorman Aggregation.
- Corollary. Any CES operator satisfies the conditions for Gorman Aggregation and thus has a representative agent.

1.11 Connection to Standard Utility

1.12 Two Period Model

1.13 Infinite Horizon

1.14 Connection to E-Z Utility