Portfolio Theory with Settlement Frictions*

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Abstract

We introduce settlement frictions into a portfolio problem. Settlement frictions arise due to the requirement of financing negative settlement balances using overnight funding obtained in an over-the-counter (OTC) market. We derive closed form solutions to the settlement costs and show how they can be encoded in a liquidity yield and asset pricing conditions. We use the framework to examine how changes in settlement frictions affect liquidity premia, the volume and dispersion of rates in the OTC market, optimal portfolios, and welfare.

Keywords: Asset Pricing, Portfolio Theory, OTC markets

JEL Classification: E51,E52, E58, G12, G21

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1. Introduction

In this paper we study a portfolio problem subject to settlement frictions. An important feature of financial markets is that many portfolio positions expose investors to sudden liquidity needs. Liquidity needs arise when investors have to unexpectedly settle portfolio positions. Examples of settlements needs include the withdrawal of deposits, the drawdown of credit lines at banks, the liquidation of shares at money-market funds, the settlement of margin calls, the realization of stock option against hedge-funds or security dealers, claims against insurance companies, etc. To meet these sudden liquidity needs, financial institutions must use settlement instruments. If an institution lacks settlement instruments, it must borrow them. In many instances, that borrowing is frictional and lead to liquidity risks.[1] Settlement needs may be uncorrelated or correlated, with asset payoffs and valuations. Together, settlement needs and the the underlying payoff risks jointly determine the required rate of return on assets. Thus, liquidity risks have implications for portfolio decisions which can give rise to different liquidity premia across assets.

We consider settlement frictions that arise from delayed transactions in over-the-counter (OTC) market frictions in the market for settlement instruments. Whereas the literature on OTC markets is vast, the focus has typically been on how the speed of the realization of gains from trade in OTC market translated into premia.[2] The focus here is different, as there is no time discounting. Instead, OTC frictions translate into a shadow cost of being short of settlement instruments and having an excess of such instruments. This gives rise to a portfolio theory where, in addition to payoff risk, assets differ in the settlement needs they create.

The model works as follows: At a given point in time, agents make portfolio decisions taking into consideration the risk in the valuation of assets and the potential settlement needs these assets can create. Then, as settlement needs are realized, there is a distribution of settlement positions. At that point, a market for settlement instruments takes place in an OTC market as the one developed in Afonso and Lagos (2015b). The novelty is that assets differ in their potential settlement needs which, in turn, lead to settlements that are cleared through borrowing in an OTC market. We think of portfolio managers as being large institutions who instructs many traders to close their settlement positions by trading in the OTC market, a feature that resembles the risk-sharing arrangement in Atkeson, Eisfeldt and Weill (2015). The idea that trade in a financial institution is carried out by many traders. Each trader, be it in surplus or deficit, then participates in a sequential OTC market with a terminal time and exogenous outside options. At the terminal time, the trader that is short must borrow at a high rate from an external party to cover the deficit. Similarly, the trader that is

[1] See for example, Ashcraft and Duffie (2007); Afonso and Lagos (2014) who study the interbank market for reserves.

[2] See Duffie, Garleanu and Pedersen (2005) for an early paper on OTC markets. The growing literature on OTC markets is surveyed by Weill (2020).

long saves its surplus at a low rate from an external party. Taking the sequence of outside options into account, agents bargain using backward induction to compute outside options.

Adapting the OTC market structure of Afonso and Lagos (2015b), using the large-trader assumption in Atkeson, Eisfeldt and Weill (2015), enables a tractable characterization of the volumes and terms of trade, for an arbitrary distribution of surplus and deficit across investors. This results because with a large-trader assumption, only the position of the institution, as a surplus or deficit institution matters for the terms of trade obtained by trader in that institution at any given round. Specializing to Leontief matching, and taking a continuous-time limit, we arrive to closed-form the volumes and terms of trade, for an arbitrary distribution of surplus and deficit across investors, at each trading round.

The closed form expressions are convenient because they make it possible to obtain an analytical liquidity-yield function that can be introduced into a dynamic quantitative model with portfolio decisions. In particular, the OTC frictions ultimately show up in a piecewise linear liquidity yield function, χ, which maps a given settlement position, s, into an marginal cost or benefit. The term $\chi(s)$ enters enters as a pricing factor in addition to the total portfolio returns. Figure 1 depicts an example of the liquidity yield function and two potential distributions of settlement needs. The asymmetry in the liquidity yield functions implies a negative penalty on those assets that generate more settlement needs when the bank is in deficit. Portfolios with greater settlement needs provoke greater losses. Mean preserving spreads in settlement needs will provoke expected losses due to the kink. By inspecting how the parameters and conditions of the OTC market impact the slopes of $\chi(s)$ we can compute how liquidity premia will respond in these markets. We can do so by assessing, in expectation, how each asset's induced settlement exposure, affects s into the liquidity yield function, and pricing the liquidity costs, using the institutions discount factor.

The theory has various applications. For example, in Bianchi and Bigio (2022) we embed the liquidity-yield function into a general-equilibrium model of banks. The withdrawal of bank deposits provokes settlement needs and we use the model to explain how the supply of central bank reserves impacts the supply of credit. Bianchi and Bigio (2022) use the function $\chi(s)$ derived here as an input. The present paper derives that function from first principles. Importantly, the slopes of that function have analytic expressions in terms of two primitive parameters and a sufficient statistic of the of the distribution of settlement needs. The two fundamental parameters are the Nash-bargaining parameter and a parameter that governs the trade efficiency. The sufficient statistic distribution of settlement needs is the market-tightness: the ratio of the absolute value of the sum all negative settlement positions relative to the ratio of all positive positions.

Beyond the appeal of providing a liquidity function that can be conveniently incorporated into a portfolio problem, by obtaining an analytic formula, we can analyze how three features: the market-

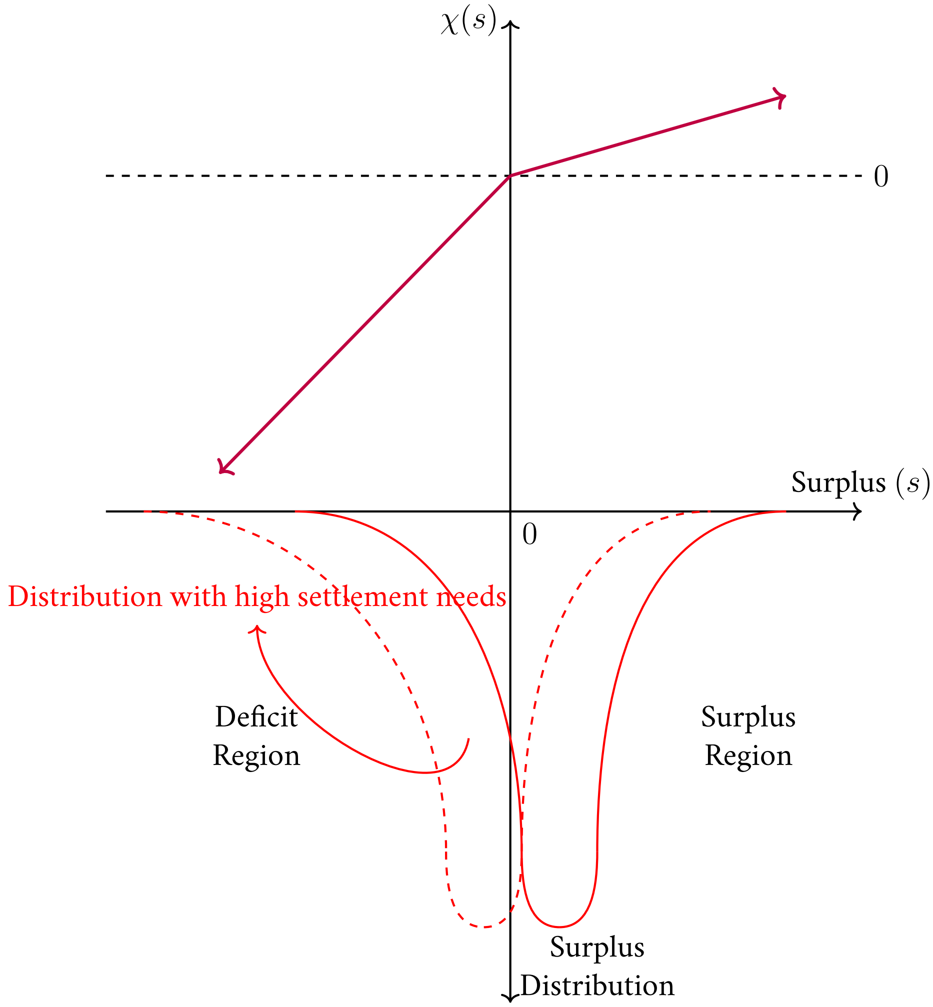


Figure 1: Liquidity Cost and Surplus Distribution

tightness and the efficiency and bargaining parameters can affect liquidity yield function. These features, in turn, impact portfolio decisions and asset liquidity premia. These comparative statics can also be used to relate liquidity premia to observable features of the OTC market In particular, we study the comparative statics the slopes of the liquidity-yield, the average rate on the OTC market, the dispersion of rates and the volume of trade. This comparative statics can be used by future studies to identify identify parameters—or shocks—to the OTC market tightness or the efficiency and bargaining parameters.

Regarding the comparative statics with respect to the market tightness, we show that a tighter market for settlement instruments increases the costs of being short of settlement instruments whereas it increases the benefit of having a settlement surplus. In Figure 1, this appears as an reduction in the slope of the positive portion of the liquidity yield function and a increase in the negative portion. Effectively, this change increase the concavity $\chi(\cdot)$, implying greater liquidity premia—-even in risk-neutral environments. Moreover, increases in the market tightness affect the average rates in the OTC market. In terms of the dispersion of rates, market tightness increases (decreases) the dispersion of rates if the market tightness is above (below) 1. Finally, the volume of trade always decreases with the market tightness when it is above, but is invariant when the tightness is below one—a result of Leontief matching.

Changes in the bargaining power of lenders have a similar effect to changes in the market tightness in that it the slopes of $\chi(\cdot)$ in the same direction and the same about rates. However, the bargaining power has no effects of the volume of trade.

A second set of results regard the trading efficiency. Whereas an increase in the market tightness increases liquidity premia, this is not true about the market's efficiency. By contrast, the effect of market the market's efficiency is non-monotone, depending on whether the market tightness is above or below one. In particular, market efficiency increases the surplus extraction of the side that is shorter: when market tightness is above one, surplus lenders are in short supply. Thus, increases in the efficiency increases the average rates. The opposite holds when the tightness is below one. Because efficiency raises the rent extraction of the short side of the market, the cost (benefits) of funding (lending) deficits (surpluses) increase with efficiency when tightness is above (below) 1. However, for the long-side of the market, the effect is ambiguous given a tension between worse terms of trade and more trading. Critically, efficiency decreases the dispersion of rates and increases the volume of trade.

These comparative statics are important because they can be potentially observed in the data, such as for example, the Federal Funds market. Our comparative statics indicate how we can identify changes in market efficiency and the dispersion of settlement needs from data on the range of rates traded in the OTC market and its volume. These comparative statics are useful to identify shocks to the market tightness and efficiency of trade, variables that are typically unobservable. Typically, these can be translated into shocks to the demand for settlement instruments. A recent burgeoning literature, (e.g., see Lopez-Salido and Vissing-Jorgenson, 2023; Afonso et al., 2023; Lagos and Navarro, 2023) is focused identifying the demand for reserves. To so, they obtain instruments for the supply of reserves, thereby obtaining estimates for shocks to the demand of reserves. The equilibrium rates in the OTC here produces such a demand curve with predictions regarding observables, discount window loans and shocks to the market tightness, that should be correlated with demand shocks.

We finish the paper with a implications of our theory by building a connection with standard portfolio theory. By taking first-order conditions from the portfolio problems, we can decompose the premia of assets with respect to risk-premia, a pure liquidity-premia that emerges regardless of risk-aversion, and the risk-pricing of liquidity premia. We show that portfolio decisions are not efficient, as investors do not internalize the effect of their portfolios on the settlement needs.[3] This feature has implications for the liquidity regulation.

Literature Review. Our work is positioned at the intersection of literature on portfolio management with frictions and OTC market studies, offering new insights into how settlement frictions and market structures influence portfolio decisions and liquidity premia.

The literature on portfolio choice with frictions study how features such as portfolio constraints or transaction costs, as for example Constantinides (1986); Basak and Cuoco (1998); Vayanos and Vila (1999); He and Krishnamurthy (2013); Brunnermeier and Sannikov (2014). Other studies have focused on uninsured risks, Constantinides and Duffie (1996); Krueger and Lustig (2010). Typically, this literature models frictions as exogenous: they correspond directly to technological parameters. Here we focus on a specific friction: the inability to sell assets and the OTC nature in the market for settlements. An important difference is that the settlement friction here depends on the individual portfolio choices. The implications differ because in GE, the relative supply of assets affects the frictions and there are externalities. Furthermore, while there is idiosyncratic risk, the settlement need is uninsured, this risk is controlled and affected by portfolios.

Parting from Duffie, Garleanu and Pedersen (2005), the literature on OTC markets studies models where assets trade in presence of search frictions. This literature has identified how features of the trading environement such as the speed of transactions, the heterogeneity in the motives for trade affect trading volumes and impact liquidity premia. Whereas the literature began with strong restrictions on portfolio holdings, namely binary holdings, work by Lagos and Rocheteau (2009) allow for arbitrary portfolio holdings bringing this literature closer to standard portfolio theory.

[3]This is true because the collections of settlement payments outside the OTC is not rebated back to investors.

Hugonnier et al. (2022) consider heterogeneity in asset selling speeds and Uslu (2019) introduced risk-averse behavior into this class of models. A key feature that distinguishes these models from hours is that trading speeds affect asset values because time is discounted, a feature that depresses the option value of selling assets when gains from trade emerge. Silva et al. (2023) go a step further and, using perturbation methods, study portfolio problems that explicitly take into account trading times when agents want to modify their portfolios.

In our model, time discounting plays no role. Moreover, the marginal cost or benefit of a settlement is constant regardless of the side of positions. The timing here is with regards to the trading within as single date. The timing of trades matter because it determines the terms of trade of a specific settlement. Our paper is rather about the correlation between the settlement needs created by each asset, and the costs of meeting those settlement needs.

In that senses, our theory can be seen as a reformulation of the liquidity asset-pricing model of Holmstrom and Tirole (2001) or the many formulations of money-search models with assets that cannot be used for trade, (see Lagos, 2010; Williamson and Wright, 2010; Lagos et al., 2017). In that paper, like here, liquidity needs arise randomly—envisioned as re-investment costs. The key difference is that liquidity can be re-allocated according to the nature of the OTC market, which can be, in principle traced to observables. A further advantage is that we incorporate our theory into a standard portfolio problem. This allows us to consider frictionless asset-pricing and risk-neutral liquidity pricing as special cases of our theory.

The OTC block of our model is very similar to Afonso and Lagos (2015a). That paper specializes the model of Afonso and Lagos (2015b) to the case where agents have $\{-1, 1\}$ settlement positions. This assumption is akin to the assumption regarding the individual traders in our model. While the matching structure is different, some of the formulas here resemble those found in Afonso and Lagos (2015a) which characterize the volumes and terms of trade. Relative to that work, a contribution of our work is to shows that by working with a large number of traders, an idea we borrow from Atkeson, Eisfeldt and Weill (2015), can be easily incorporated to portfolio theory.[4] A second contribution is that we derive comparative statics with respect to the dispersion of rates, outside funding, and the slopes of the liquidity yields. These features allow us to link moments regarding the dispersion in rates and outside funding to liquidity premia.

There are multiple applications of our theory, beyond the one in Bianchi and Bigio (2022), described above. Other papers such as Bigio and Sannikov (2019); Arce et al. (2019); Piazzesi et al. (2019) Bianchi et al. (2020) study a risk-neutral investor with settlement needs in dollars and domestic currency to analyze the effects of liquidity frictions on exchange rates and interest-parity deviations. investigate the role of liquidity frictions for the transmission of monetary policy in the

[4]A similar big family assumption in search models is also present in Shi (1997).

new-Keynesian model. Lenel et al. (2019) use a similar liquidity friction to study the short-term rate puzzle and its macroeconomic implications.

2. Portfolio Problem with Settlement Costs

2.1 Environment

We present a dynamic partial equilibrium model where investors take portfolio positions in assets and liabilities. These assets vary in payoffs, but importantly, create different settlement needs. For simplicity, we assume that payouts and settlements are in real terms.

Timing. Time is discrete, indexed by t, and there is an infinite horizon. The decisions effectively collapse to a two period model, but for now, we present the model in full generality. Each period has two stages: a *portfolio stage*, and a *settlement stage*. In the portfolio stage, investors take portfolio positions, considering the asset returns and possible liquidity/cash needs. At the beginning of the settlement stage, investors are subject to shocks to their cash positions depending on the assets they hold and the realization of their payouts. At the settlement point, investors must settle their cash-flows shocks. They can do so by borrowing in an OTC market—for unsecured—credit or with a lender of last resort. We can interpret the lender of last resort as be a central bank—if we conceive banks as investors—or current account borrowing—if we think of a non-banking sector that borrows from the banking system.

Asset Space. There is a collection of assets, $\mathbb{I}=\{1, 2, ..., I\}$. Assets can be shorted, an operation that creates a liability position. Assets differ in their realized returns, but also in settlement date. Aside from the collection of assets $\mathbb{I}$, there is a special asset, m, that is used to settle all transactions—we can think of this assets as a form of money, specifically central bank reserves if we think of banks, or deposits accounts if we think of financial institutions. Finally, there are different claims on the settlement instrument corresponding to OTC credits and lines of credit from a lender of last resort—which could be a central bank or large banks lending to financial institutions.

Aggregate State. The aggregate state is X_t, which affects the return of assets and is realized at the beginning of the portfolio stage.

2.2 Investor Decisions

We model the problem of an individual investor first, and then proceed to describe the settlement frictions that ultimately induce a liquidity yield.

Portfolio Stage. In this first sub-stage, the portfolio decisions of an individual investor takes place. The investor enters the period with a given amount of wealth e_t. His wealth is composed of a portfolio of assets, $\{a_t^i\}_{i\in\mathbb{I}}$, settlements instruments, m_t, loans to other investors, f_t , and loans obtained from the lender of last resort w_t—loans to other investors are liabilities when f_t is negative.

Given the inherited portfolio, the wealth of the investor is given by:

$$e_t = \sum_{i\in\mathbb{I}} a_t^i R_t^i (X_t) + m_t R_t^m (X_t) - \bar{R}_t^f (X_t) f_t - R_t^w w_t, \tag{1}$$

where $R_t^i (X_t)$ denotes the realized gross real return on asset i. The rates R_t^m is the predetermined return on the settlement instrument, which is assumed to be a safe asset. The return R_t^w is the cost of borrowings from the lender of last resort—we assume $R_t^w \geq R_t^m$. These rates are fixed. Finally, the OTC market settlement loans features multiple rates. However, the investor earns a weighted average rate $\bar{R}^f (X_t)$, to be described below, among various OTC trades which it diversifies. Following the usual convention, we use capital R^i to denote gross rates and r^i to denote the net rates, and the same for $\bar{R}_t^f$ and R_t^w. We use lower-case letter to express the corresponding net rate of return—i.e., $r_t^i (X_t) \equiv R_t^i (X_t) - 1$.

Knowing e_t,the investor chooses the real dividend—an equity injection if negative—, c_t, and a new portfolio $\left\{\{\tilde{a}_t^i\}_{i\in\mathbb{I}}, m_t\right\}$. We use $\tilde{x}_{t+1}$ to denote a portfolio variable chosen in the portfolio stage, while x_{t+1} will denote the end-of period portfolio variable, after the settlements are made. The budget constraint is:

$$c_t + \sum_{i\in\mathbb{I}} \tilde{a}_{t+1}^i + \tilde{m}_{t+1} = e_t.$$

We impose the following linear constraint in the portfolio: Let A_t be the vector of asset holdings at the end of the portfolio stage, with the i-th entry corresponding to the i-th asset and the $I+1$ entry corresponding to the cash holdings. We impose the following:

$$\Gamma_t \cdot A_t \geq 0.$$

The matrix Γ can be chosen to impose leverage, non-negativity constraints, liquidity requirements, etc. In particular, Γ must always guarantee $m \geq 0$. We introduce these constraints to differentiate limited arbitrage, risk-premia, with liquidity premia, the novel aspect of our theory.

Settlements. Investors enter the settlement stage with the portfolio $\left\{\{a_t^i\}_{i\in\mathbb{I}}, m_t\right\}$. In the settlement stage, assets mature, possibly randomly (we can interpret shocks as random funding shocks) to the stock of investors' liabilities, or possibly negative returns on short-long positions. To settle their position and maintain a positive balance of settlements instruments, investors trade in an over the counter (OTC) market.

At the beginning of the settlement stage, each asset experiences a shock ω_t^i to its outstanding balance. Given this shock, end-of-period balance of a_{t+1}^i is given by the law of motion:

$$a_{t+1}^i = \tilde{a}_{t+1}^i (1 + \omega_t^i). \tag{2}$$

When ω_t is positive we interpret the shock as a positive funding shock where the investor must be paid by other investors or institutions. When the shock is negative we interpret the shock as debit. In particular, the fraction $-\omega_t^i$ matures. The payouts ω^i follows a cumulative distribution $\Phi^i(\cdot)$.

Importantly, the shocks lead to different settlement positions. We define the settlement balance as:

$$s\left(\left\{a_t^i\right\}_{i\in\mathbb{I}}, m_t\right) = \underbrace{\tilde{m}_{t+1} + \sum_{i\in\mathbb{I}} \frac{R_{t+1}^i (X_{t+1})}{R_{t+1}^m} \omega_t^i \tilde{a}_t^i}_{\text{settlement balance}}. \tag{3}$$

The surplus s is produced by the portfolio $\left\{\left\{a_t^i\right\}_{i\in\mathbb{I}}, m_t\right\}$ and the shocks ω. The first term is the cash balance which is the cash held from the portfolio stage. The second terms represents the accounting of cash balances realized when the payment needs are realized. From (3), it follows that if a bank faces a large-enough payment shock, the level of cash falls below the zero.

To re-balance portfolios and satisfy that $m_{t+1} \geq 0$ investors must lend/borrow reserves in an OTC market f_{t+1} or resort to the central bank's discount window for additional reserves, w_t. By convention, we interpret f_{t+1} as funds lent out and w_{t+1} as borrowed funds, hence the sign of interest compensation in (1).

Taking this borrowing and lending into account, reserves evolve from the settlement stage to the next portfolio stage as follows:

$$m_{t+1} = \underbrace{\tilde{m}_{t+1} + \sum_{i\in\mathbb{I}} \frac{\mathbb{E}\left[R_{t+1}^a\right]}{R_{t+1}^m} \omega_t^i \tilde{a}_t^i}_{s\left(\left\{a_t^i\right\}_{i\in\mathbb{I}}, m_t\right)} + f_{t+1} + w_{t+1}. \tag{4}$$

That is, the end-of-period cash, m_{t+1}, is given by the amount of reserves after the shock minus the reserves lend in the OTC market plus the reserves borrowed from the lender of last resort.

Importantly, and this is the key novelty in this paper, $m_{t+1} \geq 0$. This assumption implies that the investor has to settle its cash balances with reserve assets, or borrowing. Ideally, if his settlement position is negative, $s\left(\left\{a_t^i\right\}_{i\in\mathbb{I}}, m_t\right) < 0$, it will want to borrow from investors in surplus, $f_{t+1} \geq 0$. However, this form of funding is short-term and unsecured, implying that the investor must search for a counter-party willing to lend. Thus, a natural way to think of this settlements is through OTC markets. Since the market is OTC, funding is not guaranteed and parts of the funding deficits must be closed through borrowing settlement instruments or, ultimately, through a credit line or a discount window.

Next, we describe how the OTC market operates and derive an important transformation to the problem. The important thing to bear in mind is that the payment shocks generate a distribution of settlement positions among investors.

2.3 Settlements

Notation. In this section we avoid time subscripts. The understanding is that settlements occur within a period t of the model. First, we summarize the main results of the section, namely how a single function summarizes the behavior of the settlement, and then move on to describe the environment in more detail. Here we recognize an individual investor with the variable j.

Key Objects. By the settlement stage, an investor j has a balance s^j of settlements. Across all investors, there's a distribution of balances. Each investor has a large number of traders that can be instructed to either lend a position of cash if they have a surplus or borrow a position if they face a deficit. The result of a search and bargaining process among the traders of a bank allows them to lend or borrow f^j of balances at an average rate $\overline{R}^f$. For an investor in deficit, the remainder of borrowing positions that are not borrowed in the OTC market, w^j, must be tapped from the lender of last resort at rate R^w. The key object of interest are $\left\{f^j, w^j, \overline{R}^f\right\}$.

In what follows, we setup the environment in a way that the corresponding amounts traded in the OTC market and borrowed from the discount window, $\{f^j, w^j\}$, are given by two endogenous probabilities (rejections) $\{\Psi^+, \Psi^-\}$. The probabilities refer to the probability that a dollar surplus is lent out in the interbank market or a dollar deficit is borrowed, respectively. In particular,

$$f^j = \begin{cases} -\Psi^- s^j & \text{if } s^j \leq 0 \\ -\Psi^+ s_t^j & \text{if } s^j > 0 \end{cases} \qquad \text{and} \qquad w^j = \begin{cases} -(1-\Psi^-)s^j & \text{if } s^j \leq 0 \\ 0 & \text{if } s^j > 0. \end{cases}$$

In turn, the average interest rate across all OTC transactions,

$$\overline{R}^f = (1-\phi)R^w + \phi R^m,$$

is given by a reduced-form bargaining power for lenders ϕ, which also depends on the dynamics of the OTC market.

Given the trading probabilities the rate on OTC transactions, we compute the average rate earned on positive and negative balances. This averages are summarized into a liquidity–yield function, $\chi : \mathbb{R} \longrightarrow \mathbb{R}$. This function is given by:

$$\chi(s^j) = \begin{cases} \chi^- s^j & \text{if } s^j \leq 0 \\ \chi^+ s^j & \text{if } s^j > 0 \end{cases},$$

where

$$\chi^- = \Psi^-(\overline{R}^f - R^m) + (1 - \Psi^-)(R^w - R^m), \qquad \chi^+ = \Psi^+(\overline{R}^f - R^m). \tag{5}$$

We can observe that $\chi(\cdot)$ is a piecewise linear function with a kink at zero and slopes given by $\{\chi^-, \chi^+\}$. The functional form of $\{\chi^-, \chi^+\}$ is intuitive. χ^- is the average cost of borrowing to finance a dollar deficit. This average is the product of the probability of borrowing in the OTC market times the cost of funding in the interbank market plus the probability of borrowing from the last resort times the cost of funding of last resort funding. Likewise, χ^+ is the average benefit of a dollar surplus which is given by the probability of lending in the OTC market times the cost of funding in the interbank market.

Manipulating expressions, we find that the liquidity-yield function satisfies:

$$\chi(s^j) = -\left(\bar{R}^f - R^m\right) f - \left(R^w - R^m\right) w,$$

an expression that reveals that this function fully encodes the exposure that asset cashflows generate, on the settlement costs.

The Investor's Problem. Once we have described the liquidity-yield function, we can return to the portfolio problem and express it recursively:

Proposition 1. *The savings-portfolio problem is:*

$$V_t(e) = \max_{\{A\}} u(c) + \beta \mathbb{E}_\omega \left[V_{t+1}(e') \right], \tag{6}$$

subject to:

$$c_t + \sum_{i \in \mathbb{I}} \tilde{a}^i_{t+1} + \tilde{m}^j_{t+1} = e, \qquad \text{(Budget Constraint)}$$

$$\sum_{i \in \mathbb{I}} a^i_t R^i_t (X_t) + m_t R^m_t + \chi_{t+1} \left(s \left(\left\{ a^i_t \right\}_{i \in \mathbb{I}}, m_t \right) \right) = e', \qquad \text{(Equity Evolution)}$$

$$\Gamma \cdot \begin{bmatrix} A \\ 1 \end{bmatrix} \geq 0 \qquad \text{(Portfolio constraints)}$$

Note that the value of wealth depends on how future wealth is affected via the liquidity cost function χ. The portfolio problem consists of choosing weights on assets $\bar{a}$ and the cash asset $\bar{m}$ to maximize the risk-adjusted return on equity, where the return on equity, R^e, is given by

$$e'\left(X_t\right) \equiv \sum_{i\in\mathbb{I}} a_t^i R_t^i\left(X_t\right) + m_t R_t^m + \chi_{t+1}\left(s\left(\left\{a_t^i\right\}_{i\in\mathbb{I}}, m_t\right)\right).$$

The Investor's Portfolio Problem. Bianchi and Bigio (2022) show that a special case of this investor's problem satisfies portfolio separation. Indeed, once the current dividend is chosen, if we substitute out loans, m_t, from the budget constraint, and suppress time subscripts, we have that the bank's objective is:

$$\Omega_t \equiv \max_{A}\left(\mathbb{E}_\omega\left[\left(R^m e + \sum_{i\in\mathbb{I}} \underbrace{\left(R^i\left(X_t\right) - R^m\right)}_{\text{Liquidity Premium}} \bar{a}_t^i + \underbrace{\chi_{t+1}\left(s\left(\left\{\bar{a}\right\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}_t^i\right)\right)}_{\text{Liquidity Yield}}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \tag{7}$$

subject to $\Gamma_t \cdot A_t \geq 0$.

An important feature of the problem is that even if the investor is risk neutral, as when $\gamma = 0$, different assets can still feature different expected-return premia. These premia are fully derived from the assets' liquidity properties and the conditions of the interbank market.

2.4 Asset Examples

In this section, we present different examples of assets whose payoff risk and liquidity risk differ. For the purpose of the examples, in each case, we assume a single asset is held in the portfolio.

Deposit Flows. In Bianchi and Bigio (2022), we consider a special case with three assets: loans (ℓ), deposits (d), and central bank reserves play the role of the settlement instrument. That paper assumes loans are risk-free, but deposits are subject to withdrawal shocks. In that case, the settlement position is given by:

$$s\left(\left\{\ell, d\right\}, m\right) \equiv m - \omega \frac{R^d}{R^m} d.$$

Deposits are assumed to be a liability in that paper, so they enter with constraint $-d = \bar{a}_t^i$ and the constraint $\bar{a}_t^i \leq 0$.

Credit Lines. Consider now the case of a credit line with pre-specified rates. We can model a credit line as follows: the credit line specifies an amount k to be drawn upon the realization of a state X_t'. To write it in a budget-neutral way, we say that the credit line is both an asset and a liability: we associate two assets such that $k = \bar{a}_t^i = -\bar{a}_t^j$ and $\left\{\bar{a}_t^i, \bar{a}_t^j\right\}$ with the credit line.

If the state is not realized, the payoff is $R^i(X_t) = R^j(X_t) = R^m$. If the state is realized, the payoffs are $R^i(X_t') > R^j(X_t')$ so the bank earns an accounting profit from the draw down of the credit line. However, the credit line creates a settlement position:

$$s\left(\left\{\bar{a}_t^i, \bar{a}_t^j\right\}, m\right) \equiv m - \omega \frac{R^j}{R^m}\bar{a}_t^i.$$

In comparison to a deposit withdrawal, the credit line is an asset whose payoffs increase in a given state, but also exposes the issuer to liquidity risk. Insurance contracts have that same feature.

Margin Calls. In the case of a deposit flow or a credit line, returns are pre-determined so the only source of risk is liquidity risk. Margin calls are assets whose returns are correlated with liquidity needs. Consider now the special case of an asset a^j bought on credit. For simplicity, let's assume the asset does not pay dividend at a given date, but will rather mature in a later period. We also normalize the price to one. A margin call specifies that for all states X_t' such that the asset falls in value below k, the creditor must bring cash up from. That is, the asset's position creates a cash need $w = \min\left\{1 - R^j(X_t'), k\right\} a^j$. In this case, the margin call creates a settlement need of:

$$s\left(\bar{a}_t^j, m\right) \equiv m - \min\left\{1 - R^j(X_t'), k\right\} a^j.$$

In the two examples above, the deposit flow or a credit line, returns are pre-determined so the only source of risk is liquidity risk. By contrast, with margin calls, the asset returns are correlated with liquidity needs. This distinction is important when we compute optimal portfolios.

Bond Coupons and Stock Dividends. In this example, we want to distinguish assets by the timing of their payoffs. Typically, asset pricing models do not make a distinction between the timing....

3. OTC Market for Settlements

In this section, we derive expressions for $\left\{\Psi^+, \Psi^-\right\}$ together with a value for the average rate on interbank loans $\overline{R}^f$. These equilibrium objects are essential to derive the liquidity-yield function, χ. For that, we need to present a mapping from primitives, and the initial distribution to the endogenous objects $\left\{\Psi^+, \Psi^-, \overline{R}^f\right\}$. The key object in our microfoundation is the market tightness in the

OTC market. Consider the sum of all negative and positive positions in the OTC market:

$$S^{-}=\int_{0}^{1}\max\left\{-s^{j},0\right\}dj \quad \text{and} \quad S^{+}=\int_{0}^{1}\max\left\{s^{j},0\right\}dj.$$

The market tightness of the OTC market $\theta\equiv S^{-}/S^{+}$. The triplet $\left\{\Psi^{+},\Psi^{-},\overline{R}^{f}\right\}$ is a function of the market tightness, θ, whose functional form depends on two primitive parameters, η and $\bar{\lambda}$. In turn, these parameters are related to a trading protocol.

The market structure of the OTC market is a sequence of two-sided over-the-counter (OTC) markets as in Afonso and Lagos (2015b). Unlike in Afonso and Lagos (2015b), investors have a large number of traders in charge of a small amount of orders, an assumption found in Atkeson et al. (2015). Like in Afonso and Lagos (2015b) investors with deficits and surpluses are segmented. Then, bankers participate in N trading rounds. In each round, a number of matches is formed. When matched, traders bargain over the rate of an OTC market loan. The parameter η reflects the bargaining power of borrowers. We work with the continuous-time limit and infinitesimal order size where the number of rounds N goes to infinity. In that limit, $\bar{\lambda}$, is the Poisson meeting intensity. This leads to an environment akin to Duffie et al. (2005) with a terminal period where traders can trade with discount window with probability 1. We present the setup in greater detail, and then specialize to obtain tractable formulations.

We proceed as follows. The next section describes the underlying trading environment when trades are of a given fixed size. We first derive a law of motion for trading probabilities. We then derive a first-order difference equation for the OTC market rate as a function of the round. Then, we develop the continuous-time limit of that difference equation, and obtain a differential equation that characterizes the OTC market rates as a function of the trading round. We then develop the properties of the solution.

3.1 General Formulation of the Afonso-Lagos Model

By the end of every portfolio stage, the settlement shocks generate a distribution of dollar deficits and surpluses across investors. If an investor has a surplus position s, it tries to lend those funds at the OTC market. If it can't lend the funds, it only earns an interest on settlement instruments R^{m}. If an investor faces a deficit, it tries to borrow those reserves in the OTC market at a rate below the discount rate, R^{w}.

Trading Orders and Trading Rounds. Positions are settled through orders. We assume that orders are of a fixed size. A fixed-size order of is an instruction to lend or borrow Δ reserves.[5] We assume that in each bank, there is one trader in charge of closing one and only one order of size Δ. Thus, a bank that wishes to lend (borrow) a surplus (deficit) of reserves can places as many lending (borrowing) orders of fixed size, Δ, as his surplus allows. In this section, we study the problem of a trader at a given bank that is aiming to allocate (borrow) Δ funds, and study the limit as $\Delta \to 0$.[6]

The assumption of fixed size orders has two roles: First, unlike money-search models, where a single entity searches for a counterpart, here each order is searched and bargained by an individual trader. We can think of large institutional investors that have multiple traders searching for counterparts and each trader tries to fulfill one order. This assumption is technically convenient because it allows to take special limits such that the overall positions of the counterparts is irrelevant—all that matters is the side of the market in which the parties are found. The assumption is also realistic. In practice, financial institutions have multiple traders operating their trading desks. It is unrealistic to think that a single trader trades can trade on behalf of an entire bank: No financial institution would want to risk its entire trading book to a single trader making a mistake (or pretending to have made a mistake).[7] Second, the fixed-size assumption simplifies the problem because, otherwise, we would have to consider the combinatorial possibilities induced by the identity of the matches.

Matching. The OTC market is open for $N-1 \geq$ sequential trading rounds. We enumerate each round by $n \in \{1, 2, ...N\}$. By convention, round N is where banks in deficit go to the discount window. In addition, we let τ denote a normalized round as $\tau = n/N$ so that $\tau \in [0, 1]$. Lending and borrowing orders are matched on each round. If a match results in an exchange, the masses of deficits and surpluses evolve depending on the amount of orders that were closed. The matching process is repeated the following round, from the remaining orders that remain open.

There is a matching function $G : \mathbb{R}^2_+ \to \mathbb{R}_+$. The arguments of G is the sum of all surplus and deficits orders and the output corresponds to a number of matches. G satisfies the following

[5] A fixed size seems a natural assumption. Typically, trades in these markets are executed in rounded amounts —e.g., 100'000 dollars, 1'000'000 dollars, etc.

[6] The idea behind the modeling of fixed sized order convention is that if a lender bank places lending orders that exceed his excess reserves, there is a chance it will not be able to have the funds to transfer to the borrowing bank. If a lending bank lacks the funds to transfer to the bank in deficit, the bank would violate a contract and face a large legal cost. default. For sufficiently high costs, no bank will ever place lending orders above the amounts they hold. Assuming that orders are of a fixed size, investors cannot place more than $\eta(s, \Delta) \equiv \lfloor |s| / \Delta \rfloor$ orders. Here, $\lfloor x \rfloor$ is the floor function understood as the largest integer not greater than x. Because investors can only place integer numbers of orders, typically, there will be a remainder of reserves that cannot be lent or borrowed at the OTC market. These residuals can be borrowed (lent) from the Fed at the discount window rated r^{DW} (excess reserve rate r^{IOR}) directly. Mathematically, this residual is $\phi(x, \Delta) = s - \lfloor |s| / \Delta \rfloor \Delta$.

[7] Of course, a richer model would provide a theory of the optimal trading size but here, the goal is to render tractability.

properties: (1) $G(\cdot,\cdot)$ is homogeneous of degree one and concave increasing on both individual arguments, (2) it is normalized to $G(1,1) = 1$, (3) it requires a positive mass on both sides to create matches: $G(0,1) = G(1,0) = 0$. Finally, it satisfies symmetry, $G(a,b) = G(b,a)$. Examples of G include all the constant elasticity of substitution functions, or the telegraphic function $2ab/(a+b)$.

In addition to G, a parameter $\lambda(N)$ captures the efficiency of the matching technology. The index N is used to take the limit as the number of rounds increases. We set $\lambda(N) \leq 1$ and we assume that $\lambda(N) \to 0$ as $N \to \infty$. We also assume that $\lambda(N)N \to \bar{\lambda}$ where $\bar{\lambda}$ is the intensity of a Poisson process. At the limit where the number of rounds goes to infinity, the matching intensity converges to a Poisson process with intensity $\bar{\lambda}G(a,b)$. Some interpretations of $\bar{\lambda}$ are that it captures counterparty trust or as a physical property of the environment (e.g., telephone versus an online platform).

The results in the following section are derived the continuous-time (rounds) limit but here we present intermediate results.[8] Define $S_0^+ \equiv S^+$ and $S_0^- \equiv S^-$, that is, the number of positions before trade. Given, $\left\{S_0^+, S_0^-\right\}$ we obtain the number of matches in a given trading rounds is given by:

$$g_n \equiv \lambda(N)\,G\left(S_{n-1}^+, S_{n-1}^-\right), \qquad n \in \{1,2,\ldots,N\}.$$

The evolution of surplus and deficit positions follow:

$$S_n^+ \equiv S_{n-1}^+ - g_n, \qquad n \in \{1,2,\ldots,N\};$$

and

$$S_n^- \equiv S_{n-1}^- - g_n, \qquad n \in \{1,2,\ldots,N\}. \tag{8}$$

Intuitively, at each round, a number of matches are realized, so the amount of surplus and deficit positions shrink with the number of matches. With these quantities, we obtain a market tightness corresponding to each round:

$$\theta_n \equiv \frac{S_n^-}{S_n^+}, \qquad n \in \{0,1,2,\ldots,N\}.$$

These formulas assume that all matches result in trade, this is something we verify later.

Next, we define the matching probabilities $\{\psi_n^+, \psi_n^-\}_{n=1}^N$, for a surplus and deficit respectively,

[8] Technically, with orders of finite size, matches are given by the function $g_n = \lfloor \lambda(N)\,G\left(M_{n-1}^+, M_{n-1}^-\right)/\Delta \rfloor \Delta$ where the floor function and multiplication by Δ take care of making the number of matches a number divisible by Δ. Observe that, $\Delta \to 0$, $g_n = \lambda(N)\,G\left(M_{n-1}^+, M_{n-1}^-\right)$. Since we work with this limit, we employ this function directly.

at each trading round as,

$$\psi_n^+ \equiv \frac{g_n}{S_{n-1}^+} = \lambda(N)\left(\frac{G\left(S_{n-1}^+, S_{n-1}^-\right)}{S_{n-1}^+}\right) = \lambda(N)G\left(1, \theta_{n-1}\right) \qquad \text{and}$$

$$\psi_n^- \equiv \frac{g_n}{S_{n-1}^-} = \lambda(N)\left(\frac{G\left(S_{n-1}^+, S_{n-1}^-\right)}{S_{n-1}^-}\right) = \lambda(N)G\left(\frac{1}{\theta_{n-1}}, 1\right) = \frac{\psi_n^+}{\theta_{n-1}}, \qquad \forall n \in \{1, 2, \ldots, N\}.$$

These are the probabilities that corresponding trader match and close their positions in round n.

Bargaining. As borrowing and lending orders are matched, each match has the potential to end as an OTC loan. If an agreement is reached, the bank placing the lending order transfers Δ reserves to the bank short of reserves. This occurs during the settlement stage, so the bank in deficit does not have to borrow Δ from the discount window. In turn, the bank lending the funds no longer earns the interest on reserves. Reserves are returned to the lender by the next portfolio stage.

At a given match, traders bargain over an OTC market rate , R_n^f. In general, each rate could be match specific and depend on the wealth of the banks matching. We will verify that as the size of trades vanishes, $\Delta \to 0$, rates only depend on the round when a given match is formed . The rate on the loan R_n^f is paid back in the next portfolio stage.

The rate R_n^f is determined trough bargaining. The outside options for borrowers and lenders varies with the matching round. If a match is reached by the N-th trading round, the outside option is 9 and for the borrower it is the discount window rate minus the rate on reserves, $R^w - R^m$. In earlier rounds, the outside option is the expected value of entering the next round with an unmatched position.

Now, consider an individual trader that bargains on behalf of his institution. The trader is responsible of closing the order of size Δ. When bargaining, traders must form an expectation of his and his trading partner's equity. These equity flows will depend on the trades of other traders in each bank. Constructing an expectation of what other traders do is an insurmountable problem to deal with. However, here we work with the limit of size orders as $\Delta \to 0$, and thus, assume that when traders bargain, they build expectations over their bank's values using the law of large numbers. Thus, each trader assumes that trades of each bank are closed at an average OTC market $\overline{r}^f$ —a rate we have to solve for. Furthermore, they build an expectation of the fraction of surpluses —or deficits— that are allocated in the OTC market as Ψ^+ and —Ψ^-.

Under this assumption of how traders build expectations, if the corresponding institution features a deficit, $s^j < 0$, the trader estimates that the fraction $\Psi^- |s^j + \Delta|$ will be borrowed from the OTC market at an average rate $\overline{R}^f$ and $(1 - \Psi^-) |s_t^j + \Delta|$ will be borrowed at the discount rate. If in turn, his bank features a surplus, $s_t^j > 0$, he expects that $\Psi^+ |s^j - \Delta|$ funds will be lent at the OTC market and that $(1-\Psi^{+})\,|s^j-\Delta|$ of the funds will remain idle.

We defined the liquidity-cost function for a bank with surplus s given by $\chi(s)$. We have to find that function, but suppose we already have it. The individual trader expects the equity of his bank to evolve according to the value of his bank investors assets tomorrow and the cost settling positions in the OTC market. The expected cost of settlements depends on the individual trader's position and his expectation of other trades which are consistent with χ. Suppose the trader matches in round n and trades Δ. Then, the trader expects his bank's equity to be:

$$e^j_\Delta = \sum_{i\in\mathbb{I}} a^i_t R^i_t\left(X_t\right) + m_t R^m_t\left(X_t\right) + \chi(s^j - \text{sign}\{s^j\}\Delta) + \text{sign}\{s^j\}(R_n - R^m)\Delta.$$

where R_n is the interbank market interest rate in round n and $\mathcal{E}^j$ the value of the bank's future assets, excluding payments from the OTC market. As long as Δ is sufficiently small, $\text{sign}\{s^j\} = \text{sign}\left\{s^j - \text{sign}\{s^j\}\Delta\right\}$. Now recall that χ is piecewise linear. Hence, as long as Δ is sufficiently small, despite the kink in Δ, we can group terms involving Δ as follows:

$$\begin{aligned}
e^j_\Delta &= \sum_{i\in\mathbb{I}} a^i_t R^i_t\left(X_t\right) + m_t R^m_t\left(X_t\right) + \chi(s^j) - \chi(\text{sign}\{s^j\}\Delta) + \text{sign}\{s^j\}(R_n - R^m)\Delta \\
&= \mathcal{E}^j + \text{sign}\{s^j\}\left(R_n - R^m - \chi^{\text{sign}\{s^j\}}\right)\Delta,
\end{aligned}$$

where

$$\mathcal{E}^j \equiv \sum_{i\in\mathbb{I}} a^i_t R^i_t\left(X_t\right) + m_t R^m_t\left(X_t\right) + \chi(s^j).$$

The expression for e^j_Δ says that an individual trader estimates the value of the equity of his bank to be $\mathcal{E}^j$, unconditional expectation of his bank's equity, but excluding his own trade which will tilt the value of the bank by, $\text{sign}\{s^j\}\left(R_n - R^m - \chi^{\text{sign}\{s^j\}}\right)\Delta$ away from the average. Thus, the trader considers what his trade will do to the value of his bank's equity. To derive these expressions we are using $\chi(\text{sign}\{s^j\}\Delta) = \chi^{\text{sign}\{s^j\}}.\text{sign}\{s^j\}\Delta$.

When bargaining, the trader must compute his contribution to the value of the bank. The value of the bank in the problem for the entire bank is $V\left(e'\right)$—suppressing the times subscript. The individual trader uses the law of large numbers to evaluate the expectation over all the trades of the bank, but is aware of his own trade. Thus, for the individual trader, his expected value after matching:

$$\begin{aligned}
J_M^{\text{sign}\{s^j\}}\left(n;\Delta\right) &\equiv \mathbb{E}\left[V\left(e'\right)|\text{match at round } n\right] \\
&= V\left(\mathcal{E}^j + \text{sign}\{s^j\}\left(R_n - R^m - \chi^{\text{sign}\{s^j\}}\right)\Delta\right),\ \forall n\in\{1,2,\ldots,N\}.
\end{aligned}$$

Before a match, the trader does not know in which round he will be matched, if at all. Let $J_U^+(n;\Delta)$ represent the value of being unmatched by round n. This value satisfies:

$$\begin{aligned} J_U^{\text{sign}\{s^j\}}(n;\Delta) &\equiv \mathbb{E}\left[V\left(e'\right)|\text{unmatch at round } n\right] \\ &= \psi_{n+1}^{\text{sign}\{s^j\}} J_M^{\text{sign}\{s^j\}}(n+1;\Delta) + \left(1-\psi_{n+1}^{\text{sign}\{s^j\}}\right) J_U^{\text{sign}\{s^j\}}(n+1;\Delta). \end{aligned}$$

This expression uses the trading probabilities we presented above. The value of an unmatched trader is the probability of matching times the value of matching its position, $\psi_{n+1}^{\text{sign}\{s^j\}} J_M^{\text{sign}\{s^j\}}(n+1;\Delta)$, plus the probability of going to the next round with an unmatched position.

Naturally, if by round N the trader is unmatched, a surplus gains nothing (it's as if $R_n = R^m$). Thus:

$$J_U^+(N;\Delta) \equiv V\left(\mathcal{E}^j - \chi^+\Delta\right).$$

Likewise, for a trader in a deficit bank in deficit, we have that:

$$J_U^-(N;\Delta) \equiv V\left(\mathcal{E}^j - \left(r^w - r^m - \chi^-\right)\Delta\right),$$

which is equivalent to saying that $r_n^f = r^w$ for the deficit trader in the last round.

With the valuations in hand, we are ready to describe the bargaining process. Upon a match, an interbank interest rate solves the Nash bargaining problem for a trade transaction Δ-size,

$$\begin{aligned} r_n^f(\Delta) = \arg\max_{r_n} &\left\{\left[\mathcal{S}_n^-(\Delta)\right]^\eta \left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta}\right\} \qquad (9) \\ \text{s.t. } \mathcal{S}_n^-(\Delta) &= V\left(\mathcal{E}^j - \left(r_n(\Delta) - r^m - \chi_n^-\right)\Delta\right) - J_U^-(n;\Delta) \\ \mathcal{S}_n^+(\Delta) &= V\left(\mathcal{E}^k + \left(r_n(\Delta) - r^m - \chi_n^+\right)\Delta\right) - J_U^+(n;\Delta), \end{aligned}$$

where the bank with deficit position is j and a bank with surplus position is k. Finding a general solution to this problem potentially depends on the distribution of $\mathcal{E}$. This is no longer the case as we take the limit $\Delta \to 0$, a limit that renders tractability.

Infinitesimal Orders. The first result states that as $\Delta \to 0$, r_n is obtained by solving a system of linear difference equations.

Proposition 1. *[Limit of Bargaining Problems] As $\Delta \to 0$, the interbank rate at round n is*

$$r_n^f \equiv r^m + (1-\eta)\,\chi_n^- + \eta\chi_n^+,$$

for all $n \in \{1, 2, ..., N\}$ *. Here,* $\chi^{+}_{N+1} = 0$ *and* $\chi^{-}_{N+1} = r^{w} - r^{m}$ *and* $\{\chi^{+}_{n}, \chi^{-}_{n}\}$ *solve:*

$$\chi^{+}_{n} = \psi^{+}_{n+1}\left(r^{f}_{n+1} - r^{m}\right) + \left(1 - \psi^{+}_{n+1}\right)\chi^{+}_{n+1} \tag{10}$$

and:

$$\chi^{-}_{n} = \psi^{-}_{n+1}\left(r^{f}_{n+1} - r^{m}\right) + \left(1 - \psi^{-}_{n+1}\right)\chi^{-}_{n+1} \tag{11}$$

for $n \in \{0, 1, ..., N\}$*.*

The proposition tells us that spread between the interbank rate and the rate on reserves, $r^{f}_{n} - r^{m}$, is given by a surplus sharing rule, $(1-\eta)\,\chi^{-}_{n+1} + \eta\chi^{+}_{n+1}$. In the expression, $\left\{\chi^{+}_{n+1}, \chi^{-}_{n+1}\right\}$ represent the average benefit and costs of lending and borrowing surplus and deficit positions in future rounds.

At this point, a natural question is how come wealth drops from the bargaining problem? Mathematically, we can divide the objective of the bargaining problem by any constant, including Δ, to obtain $[\mathcal{S}^{-}_{n}(\Delta)/\Delta]^{\eta}\,[\mathcal{S}^{+}_{n}(\Delta)/\Delta]^{1-\eta}$. Now, Observe that the limits

$$\lim_{\Delta \downarrow 0} \frac{V\left(\mathcal{E}^{j} + \text{sign}\{s^{j}\}\left(r_{n}(\Delta) - r^{m} - \chi^{\text{sign}\{s^{j}\}}\right)\Delta\right) - V(\mathcal{E}^{j})}{\Delta} = V'\left(\mathcal{E}^{j}\right)\text{sign}\{s^{j}\}\left(r_{n} - r^{m} - \chi^{\text{sign}\{s^{j}\}}\right),$$

by definition of derivative and using the chain rule. Using this identity, we can show that,

$$\lim_{\Delta \downarrow 0}\left[\mathcal{S}^{-}_{n}(\Delta)/\Delta\right]^{\eta}\left[\mathcal{S}^{+}_{n}(\Delta)/\Delta\right]^{1-\eta} = V'\left(\mathcal{E}^{j}\right)^{\eta} V'\left(\mathcal{E}^{j}\right)^{1-\eta}\max_{r_{n}-r^{m}}\left[\chi^{-}_{n+1} - (r_{n} - r^{m})\right]^{\eta}\left[(r_{n} - r^{m}) - \chi^{+}_{n+1}\right]^{1-\eta}.$$

Thus, as the size of transactions shrinks relative to the bank's overall position, the interbank rate is independent of the wealth of the banks that are matched. Intuitively, as trades shrink, the trade has a marginal impact on wealth, so wealth effects are second order. In other words, the only thing that matters is the trading round, but not the identity of the counterparties.

A by product of the proposition yields an algorithm to solve the interbank rate of each round: First, we can solve $\{\psi^{+}_{n}\}$ and $\{\psi^{-}_{n}\}$ forward, taking θ_{0} as given. Then, solve $\{\chi^{+}_{n}\}$ and $\{\chi^{-}_{n}\}$ backwards and obtain r^{f}_{n} as the interest via (10) and (11). Next, we confirm that the solutions χ^{+}_{0} and χ^{-}_{0}, are indeed the slopes of the liquidity cost function, χ, and furthermore, that $\bar{r}^{f}$, that defines the liquidity-cost function $\chi(s)$.

Proposition 2. *[Verification] The probability of closing deficit and surplus positions in the OTC market* *are given by:*

$$\Psi^{-} \equiv 1-\prod_{n=1}^{N}\left(1-\psi_{n}^{-}\right) \text{ and } \Psi^{+} \equiv 1-\prod_{n=1}^{N}\left(1-\psi_{n}^{+}\right).$$

The solutions to (10) and (11) satisfy:

$$\begin{aligned} \chi^{-} &= \chi_0^{-} = \Psi^{-}(\overline{r}^{f}-r^{m})+(1-\Psi^{-})(r^{w}-r^{m}) \\ \text{and } \chi^{+} &= \chi_0^{+} = \Psi^{+}(\overline{r}^{f}-r^{m}). \end{aligned}$$

where $\overline{r}^{f}$ is the average of $\left\{r_n^f\right\}$ weighted by the trade volume.

The proposition tells us how the liquidity cost function emerges as the trade sizes converge to zero. All we need to do is solve for $\left\{\chi_0^-, \chi_0^+\right\}$ which were described above. This proposition guarantees the internal consistency of the model.

Next, we derive the limit as the number of rounds goes to infinity. This limit is useful because it yields a differential form of $\left\{\chi_0^-, \chi_0^+\right\}$, that with specific assumptions about G, yields an analytic expression. We develop this continuous-time limit next.

Infinite Rounds and Continuous-Time Limit. We now study the limit where $N \to \infty$. As noted before, $\lambda(N) = \bar{\lambda}/N$, so $\lambda(N)$ converges to $\bar{\lambda}$ and, thus, is consistent with a Poisson intensity. We first derive the evolution of the masses of surpluses and deficits. The time interval between is $1/N$, which will shrink to zero. Also we index the normalized round by $\tau \in \{0, 1/N, 2/N, ..., 1\}$. Thus, as $N \to \infty$, we can associate a round with a fraction $\tau \in [0,1]$. Thus, we index all equilibrium variables by τ instead of by n. We have the following:

Lemma 1. *[Continuous Matching Probabilities] Let the number of trading rounds $N \to \infty$. Then, by the round indexed by τ, the ratio of deficit to surpluses θ_τ satisfies the following first-order homogeneous differential equation:*

$$\frac{\dot{\theta}_\tau}{\theta_\tau} = \frac{\dot{\theta}_\tau}{\theta_\tau} = -\bar{\lambda}\left[G\left(1/\theta_\tau, 1\right) - G\left(1, \theta_\tau\right)\right] = \bar{\lambda}\left[G\left(1, \theta_\tau\right) - G\left(1/\theta_\tau, 1\right)\right] \tag{12}$$

where $\theta_0 = S_0^-/S_0^+$. The corresponding matching intensities are:

$$\psi_\tau^+ = \bar{\lambda}G\left(1, \theta_\tau\right) \text{ and } \psi_\tau^- = \bar{\lambda}G\left(1/\theta_\tau, 1\right).$$

This intermediate result is useful. Once we solve ψ_τ^+ and ψ_τ^- in $\tau \in [0,1]$, just as in the case with finite rounds, we solve the differential equation for $\{\chi_\tau^+, \chi_\tau^-\}$. We have a closed form for $\{\chi_\tau^+, \chi_\tau^-\}$ as functions ψ_τ^+ and ψ_τ^-.

Proposition 3. *[Limit of Bargaining Problems] Let the number of trading rounds $N \to \infty$. Then, the solution to $\{\chi_\tau^+, \chi_\tau^-\}$ is:*

$$\chi_\tau^+ = (r^w - r^m)(1-\eta)\int_\tau^1 \psi_\tau^+ \exp\left[\int_y^1 -\left[(1-\eta)\psi_x^+ + \eta\psi_x^-\right] dx\right] dy$$

and for the expected cost of deficits:

$$\chi_\tau^- = (r^w - r^m)\left(1 - \eta\int_\tau^1 \psi_\tau^- \exp\left[\int_y^1 -\left[(1-\eta)\psi_x^+ + \eta\psi_x^-\right] dx\right]\right) dy,$$

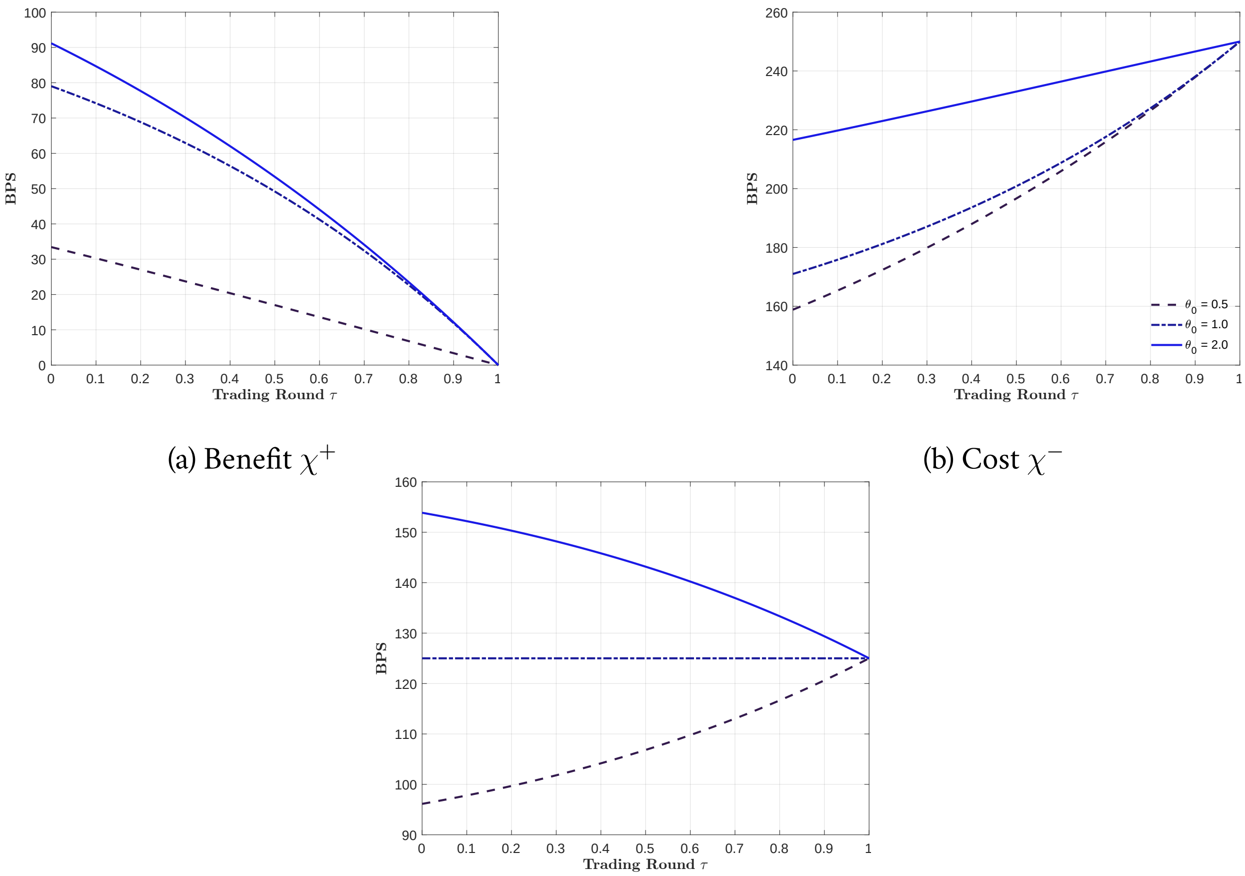
for all $\tau = [0,1]$. The slopes of the liquidity cost function are given by $\chi^+ = \chi_0^+$ and $\chi^- = \chi_0^-$.

This formula delivers $\{\chi^+, \chi^-\}$ in terms of the solution in Proposition 1. The liquidity yield is valid for any matching function G which is homogeneous of degree 1. Unfortunately, a general solution to (12) does not exist, except for special parametric forms.

The simple formulas in this proposition conceal rich dynamics for the OTC market. Like in Afonso and Lagos (2015b), market tightness varies during a trading session, depending on the matching technologies. As a result, the interbank rate varies during a trading session. Unlike the Afonso and Lagos (2015b), the infinitesimal trader's assumption renders a unique interbank rate each instant. This is because wealth effects vanish here.

Figure 2 presents the OTC market predictions of this model for a particular calibration, and for particular initial conditions. The figure reports the movement of the OTC market rate along a trading session. This is shown together with the outside options for the short and long sides of the market. As the trading session ends, the outside options collapses to zero and the spread between the discount window rate and the rate on reserves. The outside options at the beginning of the trading session yield the slopes of the liquidity cost function.

Example: Cobb-Douglas Matching. Only a limited number of matching functions G render closed form solutions. One such matching function is the Cobb-Douglas function with coefficient $1/2$. However, in the Cobb-Douglas case, with a finite T, the short-side of the market may disappear by some $\tau \leq T$. Thus, the analytic solution in the symmetric Cobb-Douglas case involves threshold market-tightness values for which the short-side of the OTC market is exhausted before the end of the trading sessions. Although this case is interesting and merits further analysis, in this paper, we turn to a special matching function that yields closed form solutions.



(a) Benefit χ^+

(b) Cost χ^-

(c) OTC Rates

Figure 2: Terms of trade throughout trading rounds

4. Analytic Solutions

We now move to a special case where $G(a,b) = \min\{a,b\}$. For this case, we obtain an analytic expression for $\left\{\chi_0^+, \chi_0^-\right\}$ in terms of θ_0. Assuming that $G(a,b) = \min\{a,b\}$, we obtain Propositions 1 in the first section. The proof involves finding a solution to the differential equations for the matching market tightness in Proposition 1 and with it, the matching probabilities. Once we obtain the solution, we solve the integrals in Proposition 3 to obtain the liquidity yield function.

Proposition 1. *Trading probabilities are given by:*

$$\Psi^+ = \begin{cases} 1 - e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1 - e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases}, \qquad \Psi^- = \begin{cases} \left(1 - e^{-\bar{\lambda}}\right)\theta^{-1} & \text{if } \theta > 1 \\ 1 - e^{-\bar{\lambda}} & \text{if } \theta \leq 1 \end{cases}.$$

Given $\theta_0 = \theta$, the tightness in the final round ($\tau = 1$) is given by:

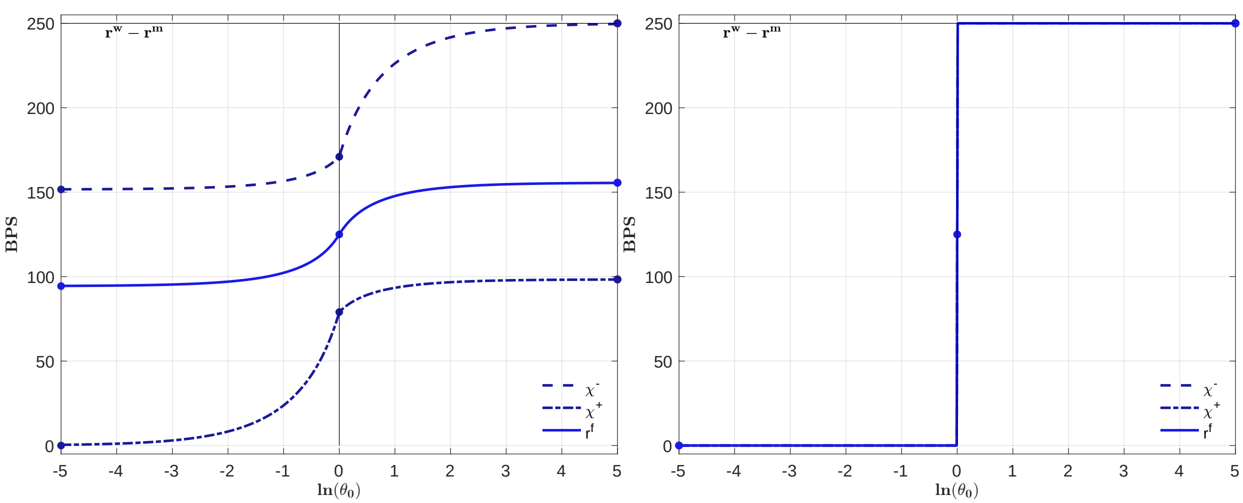
$$\bar{\theta} = \begin{cases} 1 + (\theta - 1)\exp\left(\bar{\lambda}\right) & \text{if } \theta > 1 \\ \left(1 + (\theta^{-1} - 1)\exp\left(\bar{\lambda}\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

The slopes of the liquidity-yield function are given by

$$\begin{aligned} \chi^{+} &= (r^{w} - r^{m})\left(\frac{\bar{\theta} - \bar{\theta}^{\eta}\theta^{1-\eta}}{\bar{\theta} - 1}\right) \in [0, r^{w} - r^{m}] \\ \text{and } \chi^{-} &= (r^{w} - r^{m})\left(\frac{\bar{\theta} - \bar{\theta}^{\eta}\theta^{-\eta}}{\bar{\theta} - 1}\right) \in [0, r^{w} - r^{m}]. \end{aligned}$$

The solution to $\{\chi^{+}, \chi^{-}\}$ is given by a continuous function in the terminal tightness, $\bar{\theta}$. In turn, the terminal tightness features is differentiable everywhere in θ—at $\theta = 1$, $\bar{\theta}$ has left and right derivatives equal to $\exp\left(\bar{\lambda}\right)$. With the liquidity yield function, we can obtain expressions for the equilibrium interbank rate. The average interbank rate is given by:

$$\overline{R}^{f} = \phi R^{m} + (1 - \phi) R^{w},$$



(a) Frictional Case (b) Walrasian Limit

Figure 3: OTC market Limit Properties

where the reduced-form bargaining parameter satisfies $\phi(\theta) \in [0,1]$ and is given by:

$$\phi(\theta) = \begin{cases} 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^{\eta}}{\bar{\theta} - \theta} & \text{if } \theta > 1 \\ \eta & \text{if } \theta = 1 \\ 1 - \frac{\theta^{-1} - \theta^{-\eta}\bar{\theta}^{\eta-1}}{\bar{\theta}^{-1} - \theta^{-1}} & \text{if } \theta < 1. \end{cases}$$

Because $\bar{\theta}$ has a formula that depends on its segments, to express $\{\chi^+, \chi^-\}$ in terms of θ without reference to $\bar{\theta}$, we would also need to represent it in segments. However, $\{\chi^+, \chi^-\}$ have a useful symmetry property that allows us to characterize it in only in the case where $\theta < 1$, and the case where $\theta > 1$, follows by symmetry. We have the following corollary:

Corollary 1. *[Symmetry] The solution to the bargaining problem satisfies the following symmetry:*

$$\chi^-(\theta;\eta) = (r^w - r^m) - \chi^+\left(\theta^{-1}; 1-\eta\right) \text{ and } \chi^+(\theta;\eta) = (R^w - R^m) - \chi^-\left(\theta^{-1}; 1-\eta\right).$$

Likewise,

$$\overline{R}^f(\theta;\eta) = R^w - \overline{R}^f\left(\theta^{-1}; 1-\eta\right).$$

Thanks to this symmetry, we can specialize the model to only one case.

Derivatives of the Liquidity Yields. For many portfolio-theory and asset-pricing applications, we are interested in comparative statics that involve derivatives of the liquidity-yield function. Next,

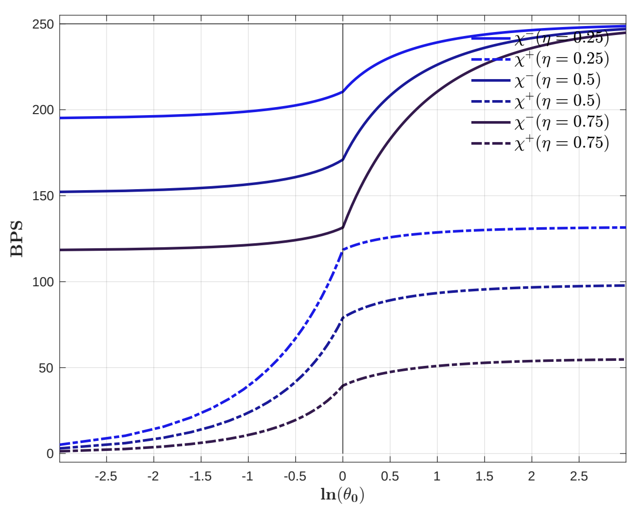


Figure 4: Symmetry in the Liquidity Yield

we derive the shape properties of χ^- and χ^+ as well as the spread between the two coefficients:

$$\Sigma \equiv \chi^- - \chi^+.$$

A first shape property of interest is with respect to the market tightness. We obtain the following result:

Proposition 4. *[Market Tightness Comparative Statics] We have the following comparative statics regarding the market tightness:*

I) Let $\theta < 1$, then $\left\{\chi^+, \chi^-, \overline{R}^f\right\}$ are all increasing and convex in θ.

II) Let $\theta > 1$, then $\left\{\chi^+, \chi^-, \overline{R}^f\right\}$ are all increasing and concave in θ.

III) Σ is decreasing and concave in θ.

The monotonicity of $\left\{\chi^+, \chi^-, \overline{R}^f\right\}$ is intuitive: the market tightness captures the relative size of settlement deficits. As deficits increase, lenders charge more for their funds and can match with greater probability. Borrowers pay higher interests on their OTC borrowings and are more likely to borrow from the last resource. The change from concavity to convexity around $\theta = 1$ follows immediately from the symmetry result we describe above. Concavity when $\theta > 1$ captures that there are decreasing returns on the extraction of surplus from the deficit side.

Next, we investigate the derivatives with respect to the efficiency of the OTC market.

Proposition 5. *[Efficiency Comparative Statics] We have the following comparative statics regarding the trading efficiency:*

I) Let $\theta < 1$, then χ^- is decreasing in λ, $\overline{R}^f$ is decreasing in λ, and χ^+ is non-montone in λ.

II) Let $\theta > 1$, then χ^+ is increasing in λ, $\overline{R}^f$ is increasing in λ, and χ^- is non-montone in λ.

III) Let $\theta = 1$, $\overline{R}^f = (1-\eta)R^w + \eta R^m$, χ^+ is increasing in λ and χ^- is decreasing in λ. In particular:

$$\chi^-_\lambda = -\eta\Psi^-_\lambda(R^w - R^m) < 0, \qquad \chi^+_\lambda = (1-\eta)\,\Psi^+_\lambda(R^w - R^m) > 0.$$

Regarding the efficiency of the OTC market, this proposition indicates that only the short side of the market has a monotone comparative static. Take for example the case of liquidity deficits that exceed surpluses. If efficiency is increased, only the surplus side earns a higher expected increase in their trades. The deficit cost of funding is ambiguous because there is a tradeoff: as efficiency increases more matches are made which reduce the cost of funding, but the cost of funding increases as a result of the dynamics of the OTC market. This result implies that we do not have monotone comparative statics with respect to the interbank market efficiency.

It is important to highlight that around $\theta = 1$ we do obtain monotone comparative statics. In particular, the liquidity yield function flattens as efficiency increasing. The intuition is that when $\theta = 1$, there are no change in the relative probabilities of matching along the trading rounds. The OTC rate is constant. The only effects on the slope coefficients of the liquidity yield occurs through the matching probabilities.

Finally, we discuss the properties of the properties regarding the bargaining power.

Proposition 6. *[Bargaining Power Comparative Statics] We have the following comparative statics regarding the bargaining power:* $\left\{\chi^{+}, \chi^{-}, \overline{R}^{f}\right\}$ *are decreasing in* η.

The result follows immediately from the fact that for any fixed path of matching probabilities, as we increase the borrower's bargaining power, rates should decline.

OTC-Volume Distribution. To derive the cross-sectional properties, we must first derive the distribution of the volume of trade in the OTC market. The starting point is to present the formula for the final market tightness generalizes. In particular:

$$\theta_{\tau} = \begin{cases} 1 + (\theta - 1)\exp\left(\bar{\lambda}\tau\right) & \text{if } \theta > 1 \\ \left(1 + (\theta^{-1} - 1)\exp\left(\bar{\lambda}\tau\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

Note that $\theta > 1 \rightarrow \theta_{\tau} > 1, \ \forall \tau \in [0,1]$, and $\theta < 1 \rightarrow \theta_{\tau} < 1, \ \forall \tau \in [0,1]$. This property is useful to characterize the distribution of the volume of trade.

Proposition 7. *Let* $S = \min\left\{S^{+}, S^{-}\right\}$. *Then, the amount of reserves traded in the interbank-market, the overall interbank market volume is given by:*

$$V = S \times (1 - \exp(-\lambda\tau)).$$

Moreover, the distribution of trading volume during trading rounds is given by:

$$v_{\tau} = \frac{g_{\tau}}{V} = \frac{\lambda \exp(-\lambda\tau)}{1 - \exp(\lambda)}.$$

Finally, the volume of discount-loans is:

$$\frac{W}{V} = \frac{\exp(-\lambda\tau) + (\theta - 1)\,\mathbb{I}_{[\theta > 1]}}{1 - \exp(-\lambda\tau)}.$$

Observe that the volume of trade has a (truncated) exponential distribution with parameter λ. Importantly, note that the volume distribution of transactions is independent of θ and only depends on λ. This is a property of the Leontief matching function. The theory predicts that most of the trading volume should occur in earlier rounds. Another implication of this Proposition is that the volume of discount window lending is useful to identify λ. Clearly, as $\lambda \to \infty$, the volume of discount loans vanish provided that $\theta \leq 1$. When $\theta > 1$, as $\lambda \to \infty$, the volume of discount loans converges to:

$$\theta - 1 = \frac{S^{-} - S^{+}}{S^{+}},$$

that is, the ratio that we would obtain if all surplus funds are lent out.

Finally, we have the following corollary regarding the volume of discount window lending relative to interbank lending:

Corollary 2. *The discount window lending relative to interbank lending decreases in the level of λ:*

$$\frac{\partial}{\partial \lambda}[W/V] = -\tau \frac{W}{V}\left(\frac{1}{1+\exp(\lambda\tau)(\theta-1)\mathbb{I}_{[\theta>1]}} + \frac{1}{\exp(\lambda\tau)-1}\right) < 0.$$

The discount window lending relative to interbank lending decreases in the level of the interbank:

$$\frac{\partial}{\partial \theta}[W/V] = \frac{W}{V}\left(\frac{\mathbb{I}_{[\theta>1]}}{\exp(-\lambda\tau)+(\theta-1)\mathbb{I}_{[\theta>1]}}\right) > 0.$$

This monotonic behavior implies that discount borrowing may be useful to identify parameters of another moment featurs the reverse monotonicity.

OTC Rate Dispersion. For many applications, we may be interested in how the dispersion of the rates in the OTC market respond to parameters. To that end, we first present the OTC market rates that result during the trading sessions. From the proofs, we obtain that the equilibrium rate at trading round $\tau \in [0,1]$ is given by:

$$r_{\tau}^{f} = r^{m} + \eta\chi_{\tau}^{-} + (1-\eta)\chi_{\tau}^{+} = r^{m} + (r^{w} - r^{m})\left(\frac{\bar{\theta} - \eta\bar{\theta}^{\eta}\theta_{\tau}^{1-\eta} - (1-\eta)\bar{\theta}^{\eta}\theta_{\tau}^{-\eta}}{\bar{\theta}-1}\right).$$

Clearly, we have that in the last trading round agents maximize the static surplus:

$$r_{1}^{f} = r^{m} + (r^{w} - r^{m})(1-\eta).$$

Therefore, since $\bar{\theta}$ is fixed rates approach the terminal rate with τ:

$$\frac{\partial}{\partial \tau}\left[r_{\tau}^{f} - r_{1}^{f}\right] = (r^{w} - r^{m})\eta(1-\eta)\left(\frac{\bar{\theta}}{\theta_{\tau}}\right)^{\eta}\frac{1-\theta_{\tau}}{\bar{\theta}-1}\frac{\partial}{\partial \tau}[\theta_{\tau}] < 0.$$

We also know that since θ_τ is increasing when $\theta > 1$, the spread is decreasing as we approach the final trading rounds and the deficit side increases relative to the surplus side. Likewise, $\theta < 1$, the spread is in increasing as we approach the final trading rounds and the deficit side decreases relative to the surplus side. T

We are interested in mapping different notions of dispersion in the interbank market to the underlying parameters. In particular, we define:

$$Q^{max-min} \equiv max_\tau \left\{r_\tau^f\right\} - min \left\{r_\tau^f\right\},$$

to be the greatest quantile difference in rates.

Proposition 8. *[Quantile Comparative Statics]*
I) Let $\theta < 1$, then $\frac{\partial Q^{\max}_{\min}}{\partial \theta} < 0$
II) Let $\theta > 1$, then $\frac{\partial Q^{\max}_{\min}}{\partial \theta} > 0$
III) Let $\theta = 1$, then $\frac{\partial Q^{\max}_{\min}}{\partial \theta} = 0$
IV) For any θ, $\frac{\partial Q^{\max}_{\min}}{\partial \lambda} \geq 0$ with equality if and only if $\theta = 1$.

This proposition characterizes the behavior of the highest dispersion in rates with respect to parameters. The main takeaway from items I through III is that dispersion in rates increases with the distance to $\theta = 1$. The intuition is that the more asymmetric the positions are, trading probabilities will change by more for the long-side of the market throughout the trading sessions. As probabilities change more, rates change more we observe more dispersion. Item IV follows from a similar logic. Trading probabilities change more, the faster the short side of the market is closing its positions. The efficiency coefficient λ does precisely that.

Figure 5 presents plots of $Q^{\max}_{\min}$ for different values of θ. The symmetric behavior around $\theta = 1$ is eviden. Likewise, the nature of increasing dispersion with greater efficiency is also evident. The implication of the results in this section is that more dispersion in interest rates can reflect more asymmetric positions. Coupled together with the earlier corollary, this proposition tells us that we can use the dispersion in the OTC rates and the ratio of discounting trading to the volume of OTC trade to identify θ and λ provided that $\theta - 1$ has the right sign.

Limiting Behavior. The virtue of a closed form is that it allows to develop comparative statics of the liquidity yield function with respect to $\{\theta, \lambda, \eta\}$. First, we describe the behavior for extreme conditions for the market tightness:

Proposition 9. *The the liquidity-yield function has the following limiting behavior:*

As $\theta \to 1$*:*

$$\chi^+ = (1-\eta)\left(1-e^{-\bar{\lambda}}\right)(r^w - r^m), \qquad \chi^- = \left(1-\eta\left(1-e^{-\bar{\lambda}}\right)\right)(r^w - r^m),$$
$$\overline{r}^f = (1-\eta)r^w + \eta r^m, \qquad \textit{and} \qquad \Psi^+ = \Psi^- = 1-e^{-\bar{\lambda}}.$$

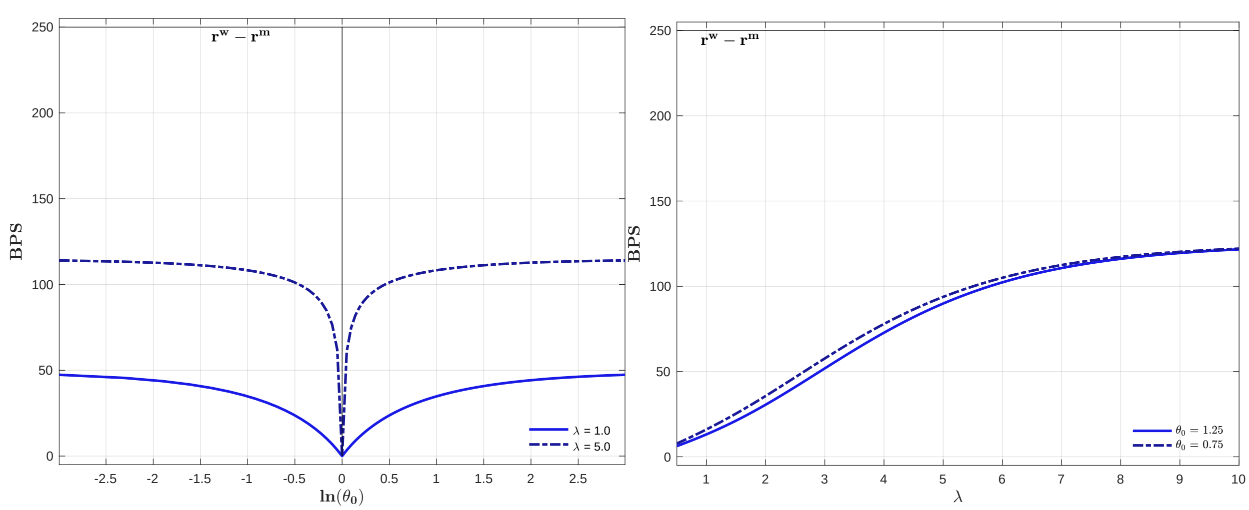
As $\theta \to \infty$*:*

$$\chi^+ = (r^w - r^m)\left(1-e^{-\bar{\lambda}(1-\eta)}\right), \qquad \chi^- = r^w - r^m$$
$$\overline{r}^f = \left(\frac{e^{\bar{\lambda}} - e^{\bar{\lambda}\eta}}{e^{\bar{\lambda}}-1}\right) r^w + \left(\frac{e^{\bar{\lambda}\eta}-1}{e^{\bar{\lambda}}-1}\right) r^m,$$
$$\Psi^+ = 1-e^{-\bar{\lambda}}, \qquad \textit{and} \qquad \Psi^- = 0.$$

As $\theta \to 0$*:*

$$\chi^+ = 0, \qquad \chi^- = (r^w - r^m)\, e^{-\bar{\lambda}\eta}$$
$$\overline{r}^f = \left(\frac{e^{\bar{\lambda}(1-\eta)}-1}{e^{\bar{\lambda}}-1}\right) r^w + \left(\frac{e^{\bar{\lambda}} - e^{\bar{\lambda}(1-\eta)}}{e^{\bar{\lambda}}-1}\right) r^m,$$
$$\Psi^+ = 0, \qquad \textit{and} \qquad \Psi^- = 1-e^{-\bar{\lambda}}.$$

As θ approaches one, matching probabilities are identical and are only a function of the meeting



(a) Varying θ_0 (b) Varying λ

Figure 5: Dispersion of OTC Rates as Functions of $\{\lambda, \theta_0\}$

technology. The interbank rate in this case is the same as if agents trade only once. As tightness approaches infinity, it is close to impossible for borrowers to close a position. Hence, they lose all of their bargaining power. The benefit of being long of reserves, χ^{+}, for this case, equals the surplus that can be extracted from the borrowing side weighted by the probability of matching and the bargaining power of the lending side. The converse logic applies as $\theta \to 0$.

Figure 3 presents the limit properties of the OTC market matching process described by the propositions above. Part 1 of the proposition establishes that as efficiency increase the OTC market rate converges to a Walrasian limit. If the banking system features an aggregate reserve deficit, $\theta > 1$, there is scarcity of reserves and the OTC market rate converges to r^{w}. In the opposite case, as $\theta < 1$, the rate converges to r^{m}. When $\theta = 1$, the interbank rate falls to an average. Part 2 establishes that as the bargaining power is entirely given to one side, the interbank rate is the outside option of the weakest side. Part 3 establishes the results as tightness goes to infinity and zero. When tightness goes to infinity, borrowing sides cost is its outside option. The benefit of being long of reserves, χ^{+}, equals the surplus that can be extracted from the borrowing side except that it is weighted by the probability of matching and the bargaining power of the lending side. A similar logic applies when $\theta \to 0$.

Next, we can characterize the comparative statics as parameters are taken to extremes.

Proposition 10. *The OTC market features the following limiting behavior:*

I) As efficiency increases, i.e. as $\bar{\lambda} \to \infty$, the outcomes of the OTC market converge to the outcomes of a Walrasian limit:

$\theta = 1$*:*

$$\begin{aligned} \chi^{+} &= \left(r^{w}-r^{m}\right)(1-\eta), \chi^{-}=\left(r^{w}-r^{m}\right)(1-\eta) \\ \bar{r}^{f} &= r^{m}\eta+r^{w}(1-\eta), \psi^{+}=\psi^{-}=1 \end{aligned}$$

$\theta > 1$*:*

$$\begin{aligned} \chi^{-} &= \left(r^{w}-r^{m}\right), \chi^{+}=\left(r^{w}-r^{m}\right) \\ \bar{r}^{f} &= r^{w}, \psi^{+}=1, \psi^{-}=\theta^{-1} \end{aligned}$$

$\theta < 1$*:*

$$\begin{aligned} \chi^{-} &= 0, \chi^{+}=0 \\ \bar{r}^{f} &= r^{m}, \psi^{+}=\theta, \psi^{-}=1. \end{aligned}$$

II) As efficiency decreases, i.e. as $\bar{\lambda} \to 0$, the outcomes of the OTC market converge to the outcomes *of a Walrasian limit:*

$$\begin{aligned}\chi^{-} &= \left(r^{w}-r^{m}\right), \chi^{+}=0 \\ \bar{r}^{f} &= r^{m}+\left(r^{w}-r^{m}\right)(1-\eta), \psi^{+}=\psi^{-}=0.\end{aligned}$$

The proposition establishes that as efficiency increases, the OTC market rate converges to a Walrasian limit. This Walrasian limit is nothing else but a version of the Poole (1970) model. As in that classic model, if the banking system features an aggregate reserve deficit, $\theta > 1$, there is scarcity of reserves and the OTC market rate converges to r^{w}. In the opposite case, as $\theta < 1$, the rate converges to r^{m}. When $\theta = 1$, the interbank rate falls to an average pinned down by η.

Next, we derive the function under the case where all the bargaining power falls on one side.

Proposition 11. *The OTC market features the following limiting behavior:*

I) As the bargaining power shifts to borrowers, $\eta \to 1$,

$$\bar{r}^{f}=r^{m},\ \chi^{-}=(1-\Psi^{-})(r^{w}-r^{m}),\ \textit{and}\ \chi^{+}=0.$$

II) As they shift to lenders $\eta \to 0$,.

$$\bar{r}^{f}=r^{w},\ \chi^{-}=(r^{w}-r^{m}),\ \textit{and}\ \chi^{+}=\Psi^{+}(r^{w}-r^{m}).$$

Part 2 establishes that as the bargaining power is given entirely to one side, that side extracts all the surplus in trades.

5. Liquidity Premia and Optimal Portfolios

Pricing conditions. As a first application of the framework, we examine the implications for the comparative statics uncovered in the previous section for asset pricing. Using the portfolio problem

(7) and assuming that X and ω are independent, we obtain:

$$\begin{aligned}
\underbrace{\mathbb{E}_X\left[R^i\right]-R^m}_{\text{asset premium}} &= \underbrace{-\frac{\mathbb{E}_{X,\omega}\left[\left(R^e_\omega\right)^{-\gamma}\left(\chi_s\frac{\partial s}{\partial a^i}\right)\right]}{\mathbb{E}_{X,\omega}\left(R^e\right)^{-\gamma}}}_{\text{liquidity premium}} \\
&\quad \underbrace{-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{conventional risk premium}} \qquad (13)\\
&= \underbrace{\underbrace{-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]}_{\text{liquidity yield}}-\underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\left(\chi_s\frac{\partial s}{\partial a^i}\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{liquidity risk premium}}}_{\text{liquidity premium}} \\
&\quad \underbrace{-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{conventional risk premium}} \qquad (14)\\
&= \underbrace{-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]}_{\text{liquidity yield}}-\underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{total risk premium}}. \qquad (15)
\end{aligned}$$

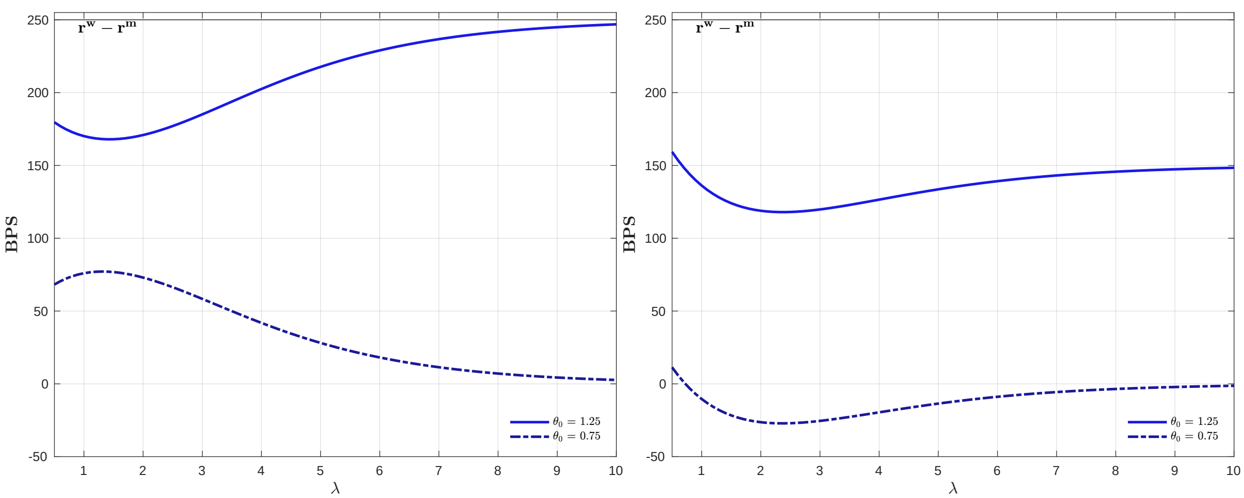
which under risk neutrality simplifies to $-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]$. The first equality unpacks the asset premium into a conventional risk premium, linking the asset's payoff to the discount factor. The first, term is the total liquidity premium: capturing the risk-adjusted expected liquidity costs of the assets. The second equation further unpacks the liquidity premium. There's a premium that emerges because the liquidity yield function is concave, so the term survives even when settlement risks are zero on average. The second term captures the covariance term between liquidity payoffs and the discount factor. The final equation, collapses the covariance terms together. It is meant to show that without considering the associated settlement needs created by an assets, the standard risk-adjustment for its payoffs is insufficient to price the risk associated with an asset.

Notice that a result derived above is that changes in matching efficiency can have ambiguous effects on that term when there is excess deficit in the market. The intuition is that while matching probabilities increase in response to an improvement in matching efficiency, the endogenous bargaining power shifts towards banks in surplus, thus raising interbank market rates. These results suggest a paradoxical result: liquidity premia may actually increase as matching improves. This feature is important because it indicates the liquidity premia are not necessarily an indication of malfunctioning in financial markets. This is not true about the market tightness, which unambiguously moves the liquidity premia.

Efficiency. We can also use our framework to understand the differences in portfolios chosen by traders in a decentralized equilibrium vis a vis a social planner that internalizes how portfolios affect market tightness and potential congestion externalities. In particular, the social planner, the problem consists of choosing portfolios such that:

$$\Omega_t \equiv \max_{A} \left(\mathbb{E}_{\omega} \left[\left(R^m e + \sum_{i \in \mathbb{I}} \left(R^i \left(X_t \right) - R^m \right) \bar{a}^i_t + \chi_{t+1} \left(s \left(\{\bar{a}\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}^i_t \right), \theta(\{\bar{a}\}_{i\in\mathbb{I}}) \right) \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}},$$

where we have replaced θ as a function of the portfolio choices, now internalized by the planner, who also assumes that X and ω are independent.



(a) No Settlement Exposure

(b) Settlement Exposure

Figure 6: Liquidity Yields and OTC Efficiency

$$
\begin{aligned}
\underbrace{\mathbb{E}_X\left[R^i\right]-R^m}_{\text{asset premium}} &= \underbrace{-\frac{\mathbb{E}_{X,\omega}\left[\left(R^e_\omega\right)^{-\gamma}\left(\chi_s\frac{\partial s}{\partial a^i}\right)\right]}{\mathbb{E}_{X,\omega}\left(R^e\right)^{-\gamma}}}_{\text{liquidity premium}}-\underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{conventional risk premium}} \\
&\quad -\underbrace{\frac{\mathbb{E}_{X,\omega}\left[\left(R^e_\omega\right)^{-\gamma}\left(\chi_\theta\frac{\partial \theta}{\partial a^i}\right)\right]}{\mathbb{E}_{X,\omega}\left(R^e\right)^{-\gamma}}}_{\text{planner internalization}} \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{\underbrace{-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]}_{\text{liquidity yield}}-\underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\left(\chi_s\frac{\partial s}{\partial a^i}\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{liquidity risk premium}}}_{\text{liquidity premium}} \\
&\quad -\underbrace{\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{conventional risk premium}} \qquad (17) \\
&\quad -\left(\underbrace{\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]+\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{planner internalization}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{-\left(\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]+\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]\right)}_{\text{liquidity yield}} \\
&\quad \underbrace{-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}}_{\text{total risk risk premium}}. \qquad (18)
\end{aligned}
$$

which also under risk neutrality simplifies to $-\left(\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]+\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]\right)$. The key distinction is that the planner now considers its effects on market tightness. This effect captures that if resources are not rebated back to investors, there is a coordination failure. By altering their portfolio holdings, investors could reduce payments to the lender of last resort.

6. Conclusions

In this paper, we introduce settlement frictions into a portfolio problem and study the implications for trading volumes, the dispersion of interest rates, and liqudity premia. In Bianchi and Bigio (2022), we build on the theory to study the interbank market and the transmission of monetary policy. However, adding supply conditions to the asset-demand conditions derived in the paper, the model can be used to study asset-pricing implications in other contexts, as we discussed throughout the draft.

There are many missing features in our OTC theory. For example, the comparative statics we derived are specific to the Leontief matching technology for which we obtain analytic expressions. As we discussed, the trade dynamics differ when we use a Cobb-Douglas technology where trade can be exhausted before a trading sessions. Furthermore, empirical evidence on interbank-markets shows that often, trading speeds up toward the end of trading sessions (see for example Wong and Zhang, 2023). Variations with time varying efficiency, search effort, or multiple offers can be used to improve the model along that direction (see Afonso and Lagos, 2014; Ashcraft and Duffie, 2007). We further abstracted from the presence of large market makers. In reality, large dealers play a central role in OTC markets and shocks to their portfolios can impact liquidity premia.

We deliberately chose an abstract setting where a single asset plays the role of a settlement instrument. In practice, we can have multiple assets playing that role, as in Bianchi et al. (2020). Likewise, there may be multiple parallel markets where different assets are used as collateral for the same settlement instruments. We can also think of layers of clearings: whereas central-bank reserves and discount-window loans are the clearing instrument and last resort for banks, bank deposits and bank credit lines serve as a clearing instrument and last resort for other segments of the finance industry. All of these dimensions can render interesting extensions.

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A. Proofs of Proposition 1 and Proposition 2

The proof of both propositions is presented in sequence. We begin with an auxiliary Lemma.

Preliminary Lemma. Market tightness follows a differential equation as described by the lemma below. With the market tightness, we obtain the matching probabilities at each round:

Let θ_0 be the initial market tightness. Then, the ratio $\{\theta_n\}$ features the following law of motion:

$$\theta_n = \theta_{n-1} \frac{(1 - \lambda(N) G(1/\theta_{n-1}, 1))}{(1 - \lambda(N) G(1, \theta_{n-1}))} \qquad \forall n \in \{1, 2, \ldots, N\}.$$

and the matching probabilities can be expressed in terms of the ratio via:

$$\psi_n^+ = \lambda(N) G(1, \theta_{n-1}) \text{ and } \psi_n^- = \lambda(N) G(1/\theta_{n-1}, 1).$$

The proof is simple. By definition and homogeneity:

$$\theta_n = \frac{S_n^-}{S_n^+} = \frac{S_{n-1}^- - g_n}{S_{n-1}^+ - g_n} = \theta_{n-1} \frac{(1 - \lambda(N) G(1/\theta_{n-1}, 1))}{(1 - \lambda(N) G(1, \theta_{n-1}))}, \qquad \forall n \in \{1, 2, \ldots, N\}.$$

where the second equality follow from the definition of g_n and uses its homogeneity property. Hence, the Lemma.

The lemma shows how we can track matching probabilities in terms of the initial market tightness. It also shows that the these probabilities are scale invariant. We use this observations in what follows.

Auxiliary Calculations Next, we describe the limit of the bargaining problem as $\Delta \to 0$. We begin with some observations:

$$\begin{aligned}
\lim_{\Delta \to 0} \left\{ \frac{J_U^+(N; \Delta) - V(\mathcal{E}^j)}{\Delta} \right\} &= \lim_{\Delta \to 0} \left\{ \frac{V(\mathcal{E}^j - \chi^+ \Delta) - V(\mathcal{E}^j)}{\Delta} \right\} \\
&= -\chi^+ \lim_{\Delta \to 0} \left\{ \frac{V(\mathcal{E}^j - \chi^+ \Delta) - V(\mathcal{E}^j)}{-\chi^+ \Delta} \right\} \\
&= -\chi^+ V'(\mathcal{E}^j); \\
\lim_{\Delta \to 0} \left\{ \frac{J_U^+(n; \Delta) - V(\mathcal{E}^j)}{\Delta} \right\} &= \lim_{\Delta \to 0} \left\{ \frac{\psi_{n+1}^+ J_M^+(n+1; \Delta) + \left(1 - \psi_{n+1}^+\right) J_U^+(n+1; \Delta) - V(\mathcal{E}^j)}{\Delta} \right\} \\
&= \lim_{\Delta \to 0} \left\{ \frac{\psi_{n+1}^+ \left(J_M^+(n+1; \Delta) - V(\mathcal{E}^j)\right) + \left(1 - \psi_{n+1}^+\right) \left(J_U^+(n+1; \Delta) - V(\mathcal{E}^j)\right)}{\Delta} \right\} \\
&= \psi_{n+1}^+ \left(i_{n+1} - r^m - \chi^+\right) V'(\mathcal{E}^j) + \left(1 - \psi_{n+1}^+\right) \lim_{\Delta \to 0} \left\{ \frac{J_U^+(n+1; \Delta) - V(\mathcal{E}^j)}{\Delta} \right\}; \\
\lim_{\Delta \to 0} \left\{ \frac{J_M^+(n; \Delta) - V(\mathcal{E}^j)}{\Delta} \right\} &= \lim_{\Delta \to 0} \left\{ \frac{V\left(\mathcal{E}^j + (r_n(\Delta) - r^m - \chi^+) \Delta\right) - V(\mathcal{E}^j)}{\Delta} \right\} \\
&= \lim_{\Delta \to 0} \left\{ r_n(\Delta) - r^m - \chi^+ \right\} \lim_{\Delta \to 0} \left\{ \frac{V\left(\mathcal{E}^j + (r_n - r^m - \chi^+) \Delta\right) - V(\mathcal{E}^j)}{(r_n(\Delta) - r^m - \chi^+) \Delta} \right\} \\
&= \left(r_n - r^m - \chi^+\right) V'(\mathcal{E}^j).
\end{aligned}$$

Similarly with the same steps we show that,

$$\lim_{\Delta\to 0}\left\{\frac{J_M^-(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\left(r_n-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right);$$

$$\lim_{\Delta\to 0}\left\{\frac{J_U^-(n;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\psi_{n+1}^-\left(i_{n+1}-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)+\left(1-\psi_{n+1}^-\right)\lim_{\Delta\to 0}\left\{\frac{J_U^-(n+1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\};$$

$$\lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\left(r^w-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right).$$

Now, let's consider the interbank rate at this limit. Consider the bargaining problem given by (9):

$$r_n^f(\Delta)=\arg\max_{r_n}\left\{\left[\mathcal{S}_n^-(\Delta)\right]^\eta\left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta}\right\}$$
$$\text{s.t. } \mathcal{S}_n^-(\Delta)=V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-J_U^-(n;\Delta)$$
$$\mathcal{S}_n^+(\Delta)=V\left(\mathcal{E}^k+\left(r_n(\Delta)-r^m-\chi^+\right)\Delta\right)-J_U^+(n;\Delta)$$

The solution to the OTC market rate doesn't change if we multiply the right-hand side of (9) by Δ. Thus, the limiting rate for round n satisfies:

$$r_n^f=\lim_{\Delta\to 0}\left\{r_n^f(\Delta)\right\}=\lim_{\Delta\to 0}\left\{\arg\max_{r_n}\left\{\frac{\left[\mathcal{S}_n^-(\Delta)\right]^\eta\left[\mathcal{S}_n^+(\Delta)\right]^{1-\eta}}{\Delta}\right\}\right\}=\arg\max_{r_n}\left\{\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_n^-(\Delta)}{\Delta}\right\}\right]^\eta\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_n^+(\Delta)}{\Delta}\right\}\right]^{1-\eta}\right\},$$

where

$$\mathcal{S}_n^-(\Delta)=V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-J_U^-(n;\Delta)$$
$$\text{and}\quad \mathcal{S}_n^+(\Delta)=V\left(\mathcal{E}^k+\left(r_n(\Delta)-r^m-\chi^+\right)\Delta\right)-J_U^+(n;\Delta).$$

Since, the solution belongs to a compact space, namely $i\in[r^m,r^w]$, this problem satisfies the conditions for the Maximum Theorem, so the continuity of the solution is guaranteed. This means that we can take limits as Δ converges to zero. Next, we by backward induction: first obtaining a solution at round N, at round $N-1$ and so forth. We do this in three steps:

Step 1: Round N-th of Matching Process Let us start in the last period of the matching process, the N-th round. In this case, the outside option limit identities of an atomistic bank in a deficit position j and an atomistic bank in a surplus position k are

$$\lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}=-\left(r^w-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)\quad\text{and}\quad\lim_{\Delta\to 0}\left\{\frac{J_U^+(N;\Delta)-V\left(\mathcal{E}^k\right)}{\Delta}\right\}=-\chi^+V'\left(\mathcal{E}^k\right).$$

Thus, the limit surpluses can be described as,

$$
\begin{aligned}
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^-(\Delta)}{\Delta}\right\} &= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-J_U^-(N;\Delta)}{\Delta}\right\}\\
&= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\
&= \lim_{\Delta\to 0}\left\{-\left(r_n(\Delta)-r^m-\chi^-\right)\right\}\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{-\left(r_n(\Delta)-r^m-\chi^-\right)\Delta}\right\}\dots\\
&- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\
&= -\left(r_n-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)+\left(r^w-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)\\
&= \left(r^w-r_n\right)V'\left(\mathcal{E}^j\right);
\end{aligned}
$$

and similarly,

$$
\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^+(\Delta)}{\Delta}\right\} = \left(r_n-r^m\right)V'\left(\mathcal{E}^k\right).
$$

Therefore, the bargaining problem for an infinitesimal size transaction in the N-th matching round can be described as

$$
\begin{aligned}
r_N^f &= \arg\max_{r_n}\left\{\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^-(\Delta)}{\Delta}\right\}\right]^\eta\left[\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_N^+(\Delta)}{\Delta}\right\}\right]^{1-\eta}\right\}\\
&= \arg\max_{r_n}\left\{\left[\left(r^w-r_n\right)V'\left(\mathcal{E}^j\right)\right]^\eta\left[\left(r_n-r^m\right)V'\left(\mathcal{E}^k\right)\right]^{1-\eta}\right\}\\
&= \arg\max_{r_n}\left\{\left[V'\left(\mathcal{E}^j\right)\right]^\eta\left[V'\left(\mathcal{E}^k\right)\right]^{1-\eta}\left[r^w-r_n\right]^\eta\left[r_n-r^m\right]^{1-\eta}\right\}\\
&= \arg\max_{r_n}\left\{\left[r^w-r_n\right]^\eta\left[r_n-r^m\right]^{1-\eta}\right\}.
\end{aligned}
$$

Taking the first order conditions, we get

$$
\eta\left(\frac{r_N^f-r^m}{r^w-r_N^f}\right)^{1-\eta} = (1-\eta)\left(\frac{r^w-r_N^f}{r_N^f-r^m}\right)^\eta.
$$

Thus, we get to the optimal interest rate

$$
r_N^f = r^m+(1-\eta)\left(r^w-r^m\right).
$$

Finally, define $\chi_N^+\equiv 0$ and $\chi_N^-\equiv r^w-r^m$. From here, we conclude that,

$$
r_N^f = r^m+(1-\eta)\chi_N^-+\eta\chi_N^+.
$$

Step 2: Round $\{N-1\}$-th of Matching Process Let now obtain a similar equations for the $\{N-1\}$-th round. Following the same steps as for the N-th round we have:

$$\begin{aligned}\lim_{\Delta\to 0}\left\{\frac{J_U^-(N-1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\} &= -\psi_N^-\left(r_N^f-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)+\left(1-\psi_N^-\right)\lim_{\Delta\to 0}\left\{\frac{J_U^-(N;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\ &= -\psi_N^-\left(r_N^f-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)-\left(1-\psi_N^-\right)\left(r^w-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)\\ &= -\left(\psi_N^-\left(r_N^f-r^m-\chi^-\right)+\left(1-\psi_N^-\right)\left(r^w-r^m-\chi^-\right)\right)V'\left(\mathcal{E}^j\right)\\ &= -\left(\psi_N^-\left(r_N^f-r^m\right)+\left(1-\psi_N^-\right)\left(r^w-r^m\right)-\chi^-\right)V'\left(\mathcal{E}^j\right)\\ &= -\left(\psi_N^-\left(r_N^f-r^m\right)+\left(1-\psi_N^-\right)\chi_N^- -\chi^-\right)V'\left(\mathcal{E}^j\right),\end{aligned}$$

and through similar steps:

$$\lim_{\Delta\to 0}\left\{\frac{J_U^+(N-1;\Delta)-V\left(\mathcal{E}^k\right)}{\Delta}\right\}=\left(\psi_N^+\left(r_N^f-r^m\right)+\left(1-\psi_N^+\right)\chi_N^+-\chi^+\right)V'\left(\mathcal{E}^k\right).$$

Define $\chi_{N-1}^-\equiv\psi_N^-\left(r_N^f-r^m\right)+\left(1-\psi_N^-\right)\chi_N^-$ and $\chi_{N-1}^+\equiv\psi_N^+\left(r_N^f-r^m\right)+\left(1-\psi_N^+\right)\chi_N^+$ so

$$\begin{aligned}\lim_{\Delta\to 0}\left\{\frac{J_U^-(N-1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\} &= -\left(\chi_{N-1}^- -\chi^-\right)V'\left(\mathcal{E}^j\right)\\ \text{and}\quad \lim_{\Delta\to 0}\left\{\frac{J_U^+(N-1;\Delta)-V\left(\mathcal{E}^k\right)}{\Delta}\right\} &= \left(\chi_{N-1}^+-\chi^+\right)V'\left(\mathcal{E}^k\right).\end{aligned}$$

Thus, the limit surpluses can be described as,

$$\begin{aligned}\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^-(\Delta)}{\Delta}\right\} &= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(i_{N-1}(\Delta)-r^m-\chi^-\right)\Delta\right)-J_U^-(N-1;\Delta)}{\Delta}\right\}\\ &= \lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(i_{N-1}(\Delta)-r^m-\chi^-\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\ &- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N-1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\ &= \lim_{\Delta\to 0}\left\{-\left(i_{N-1}(\Delta)-r^m-\chi^-\right)\right\}\lim_{\Delta\to 0}\left\{\frac{V\left(\mathcal{E}^j-\left(i_{N-1}(\Delta)-r^m-\chi^-\right)\Delta\right)-V\left(\mathcal{E}^j\right)}{-\left(i_{N-1}(\Delta)-r^m-\chi^-\right)\Delta}\right\}\\ &- \lim_{\Delta\to 0}\left\{\frac{J_U^-(N-1;\Delta)-V\left(\mathcal{E}^j\right)}{\Delta}\right\}\\ &= -\left(i_{N-1}-r^m-\chi^-\right)V'\left(\mathcal{E}^j\right)+\left(\chi_{N-1}^- -\chi^-\right)V'\left(\mathcal{E}^j\right)\\ &= \left(\chi_{N-1}^- -\left(i_{N-1}-r^m\right)\right)V'\left(\mathcal{E}^j\right);\end{aligned}$$

and, thus,

$$\lim_{\Delta\to 0}\left\{\frac{\mathcal{S}_{N-1}^+(\Delta)}{\Delta}\right\}=\left(\left(i_{N-1}-r^m\right)-\chi_{N-1}^+\right)V'\left(\mathcal{E}^k\right).$$

Therefore, the bargaining problem for an infinitesimal size transaction in the $\{N-1\}$-th matching round can be described as

$$\begin{aligned}
i^f_{N-1} &= \arg\max_{i_{N-1}} \left\{ \left[\lim_{\Delta\to 0} \left\{ \frac{\mathcal{S}^-_{N-1}(\Delta)}{\Delta} \right\} \right]^{\eta} \left[\lim_{\Delta\to 0} \left\{ \frac{\mathcal{S}^+_{N-1}(\Delta)}{\Delta} \right\} \right]^{1-\eta} \right\} \\
&= \arg\max_{i_{N-1}} \left\{ \left[\left(\chi^-_{N-1} - (i_{N-1} - r^m) \right) V'\left(\mathcal{E}^j\right) \right]^{\eta} \left[\left((i_{N-1} - r^m) - \chi^+_{N-1} \right) V'\left(\mathcal{E}^k\right) \right]^{1-\eta} \right\} \\
&= \arg\max_{i_{N-1}} \left\{ \left[V'\left(\mathcal{E}^j\right) \right]^{\eta} \left[V'\left(\mathcal{E}^k\right) \right]^{1-\eta} \left[\chi^-_{N-1} - (i_{N-1} - r^m) \right]^{\eta} \left[(i_{N-1} - r^m) - \chi^+_{N-1} \right]^{1-\eta} \right\} \\
&= \arg\max_{i_{N-1}} \left\{ \left[\chi^-_{N-1} - (i_{N-1} - r^m) \right]^{\eta} \left[(i_{N-1} - r^m) - \chi^+_{N-1} \right]^{1-\eta} \right\}.
\end{aligned}$$

Taking the first-order conditions, we get

$$\eta \left(\frac{i_{N-1} - r^m - \chi^+_{N-1}}{\chi^-_{N-1} - i_{N-1} - r^m} \right)^{1-\eta} = (1-\eta) \left(\frac{\chi^-_{N-1} - i_{N-1} - r^m}{i_{N-1} - r^m - \chi^+_{N-1}} \right)^{\eta}.$$

Finally, the solution to the interest rate is:

$$i^f_{N-1} = r^m + (1-\eta)\chi^-_{N-1} + \eta\chi^+_{N-1}.$$

Step 3: Round $\{N-2\}$-th of Matching Process Let's now study the matching process, at the $\{N-2\}$-th round. In this case, the outside option limit identities of an atomistic bank in a deficit position j and an atomistic bank in a surplus position k are

$$\lim_{\Delta\to 0} \left\{ \frac{J^-_U(N-2;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta} \right\} = -\left(\psi^-_{N-1} \left(i^f_{N-1} - r^m \right) + \left(1 - \psi^-_{N-1}\right) \chi^-_{N-1} - \chi^- \right) V'\left(\mathcal{E}^j\right),$$

and

$$\lim_{\Delta\to 0} \left\{ \frac{J^+_U(N-2;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta} \right\} = \left(\psi^+_{N-1} \left(i^f_{N-1} - r^m \right) + \left(1 - \psi^+_{N-1}\right) \chi^+_{N-1} - \chi^+ \right) V'\left(\mathcal{E}^k\right).$$

Define $\chi^-_{N-2} \equiv \psi^-_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1 - \psi^-_{N-1}\right)\chi^-_{N-1}$ and $\chi^+_{N-2} \equiv \psi^+_{N-1}\left(i^f_{N-1} - r^m\right) + \left(1 - \psi^-_{N-1}\right)\chi^+_{N-1}$ so

$$\begin{aligned}
\lim_{\Delta\to 0} \left\{ \frac{J^-_U(N-2;\Delta) - V\left(\mathcal{E}^j\right)}{\Delta} \right\} &= -\left(\chi^-_{N-2} - \chi^-\right) V'\left(\mathcal{E}^j\right) \\
\text{and} \qquad \lim_{\Delta\to 0} \left\{ \frac{J^+_U(N-2;\Delta) - V\left(\mathcal{E}^k\right)}{\Delta} \right\} &= \left(\chi^+_{N-2} - \chi^+\right) V'\left(\mathcal{E}^k\right).
\end{aligned}$$

Thus, the limit surpluses can be described as,

$$\lim_{\Delta\to 0} \left\{ \frac{\mathcal{S}^-_{N-2}(\Delta)}{\Delta} \right\} = \left(\chi^-_{N-2} - (i_{N-2} - r^m) \right) V'\left(\mathcal{E}^j\right);$$

and

$$\lim_{\Delta \to 0} \left\{ \frac{\mathcal{S}^{+}_{N-2}(\Delta)}{\Delta} \right\} = \left((i_{N-2} - r^m) - \chi^{+}_{N-2} \right) V' \left(\mathcal{E}^k \right).$$

Therefore, the bargaining problem in round $\{N-2\}$-th is,

$$\begin{aligned} i^f_{N-2} &= \arg\max_{i_{N-2}} \left\{ \left[\lim_{\Delta \to 0} \left\{ \frac{\mathcal{S}^{-}_{N-2}(\Delta)}{\Delta} \right\} \right]^{\eta} \left[\lim_{\Delta \to 0} \left\{ \frac{\mathcal{S}^{+}_{N-2}(\Delta)}{\Delta} \right\} \right]^{1-\eta} \right\} \\ &= \arg\max_{i_{N-2}} \left\{ \left[\chi^{-}_{N-2} - (i_{N-2} - r^m) \right]^{\eta} \left[(i_{N-2} - r^m) - \chi^{+}_{N-2} \right]^{1-\eta} \right\}. \end{aligned}$$

Taking the first-order conditions, we obtain:

$$i^f_{N-2} = r^m + (1-\eta)\chi^{-}_{N-2} + \eta \chi^{+}_{N-2}.$$

Step 4: Round n-th of Matching Process We continue by induction, to obtain:

$$\chi^{-}_{n} = \psi^{-}_{n+1} \left(i^f_{n+1} - r^m \right) + \left(1 - \psi^{-}_{n+1} \right) \chi^{-}_{n+1} \qquad \text{and} \qquad \chi^{+}_{n} = \psi^{+}_{n+1} \left(i^f_{n+1} - r^m \right) + \left(1 - \psi^{+}_{n+1} \right) \chi^{+}_{n+1}. \quad (19)$$

Furthermore, the optimal interbank interest rates that solve the bargaining problem at the n-th matching round is

$$r^f_n = r^m + (1-\eta)\chi^{-}_{n} + \eta \chi^{+}_{n}.$$

This, and the previous recursions are the expressions in Proposition 1. Next, we verify the consistency of the solution.

Proof of Proposition 2. The probability of matching in one of the N matching rounds for individual surplus and deficit traders are:

$$\Psi^{+} = \sum_{n=1}^{N} \psi^{+}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{+}_{m} \right) \right] = 1 - \left[\prod_{m=1}^{N} \left(1 - \psi^{+}_{m} \right) \right] \qquad \text{and} \qquad \Psi^{-} = \sum_{n=1}^{N} \psi^{-}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{-}_{m} \right) \right] = 1 - \left[\prod_{m=1}^{N} \left(1 - \psi^{-}_{m} \right) \right],$$

where $\psi^{+}_{0} = \psi^{-}_{0} = 0$. Define the weights $\{\varkappa_n\}_{n=1}^{N}$ as the distribution of matching in round n conditional on matching,

$$\varkappa^{+}_{n} \equiv \frac{\psi^{+}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{+}_{m} \right) \right]}{\Psi^{+}} \qquad \text{and} \qquad \varkappa^{-}_{n} \equiv \frac{\psi^{-}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{-}_{m} \right) \right]}{\Psi^{-}}.$$

The numerator corresponds to the unconditional probability of a match at round n and the denominator is the probability of matching at all. By the law of large numbers, this is proportional to the volume at that round. Clearly the weights sum to one. Next, we show that conditional distributions are the same for deficits and surpluses:

$$\begin{aligned} \varkappa^{+}_{n} &\equiv \frac{\psi^{+}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{+}_{m} \right) \right]}{\Psi^{+}} = \frac{\psi^{+}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \lambda(N) G(1/\theta_m, 1) \right) \right]}{\Psi^{+}} \\ &= \frac{\psi^{+}_{n} \left[\prod_{m=1}^{n-1} \frac{\theta_{m=1}}{\theta_m} \left(1 - \lambda(N) G(1, \theta_m) \right) \right]}{\Psi^{+}} = \frac{\theta^{-1}_{n-1} \psi^{+}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \lambda(N) G(1, \theta_m) \right) \right]}{\theta^{-}_{0} \Psi^{+}} \\ &= \frac{\psi^{-}_{n} \left[\prod_{m=1}^{n-1} \left(1 - \psi^{-}_{n} \right) \right]}{\Psi^{-}} = \varkappa^{-}_{n}. \end{aligned}$$

where we used **??** and the definition of ψ_n^+ and ψ_n^-.

Thus, the average interbank-interest rate is the weighted average of the interbank interest rates of each round,

$$
\begin{aligned}
\overline{r}^f &= \sum_{n=1}^{N} \varkappa_n^+ r_n^f \\
&= \sum_{n=1}^{N} \varkappa_n^+ \left(r^m + (1-\eta)\chi_n^- + \eta\chi_n^+\right) \\
&= \left[\sum_{n=1}^{N} \varkappa_n^+\right] r^m + \left[\sum_{n=1}^{N} \varkappa_n^+ \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)\right] \\
&= r^m + \left[\frac{\sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)}{\Psi^+}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Psi^+ \left(\overline{r}^f - r^m\right) &= \sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right) \\
&= \sum_{n=1}^{N} \psi_n^+ \left[\prod_{m=1}^{n-1} \left(1-\psi_m^+\right)\right] \left(r_n^f - r^m\right) \\
&= \chi_0^+.
\end{aligned}
$$

where the last line follows from solving χ_0^+ in (19) forward. This verifies that $\chi_0^+ = \chi^+$.

Similarly, if we follow the same steps we have that:

$$
\begin{aligned}
\overline{r}^f &= \sum_{n=1}^{N} \varkappa_n^- r_n^f \\
&= r^m + \left[\frac{\sum_{n=1}^{N} \psi_n^- \left[\prod_{m=1}^{n-1} \left(1-\psi_m^-\right)\right] \left((1-\eta)\chi_n^- + \eta\chi_n^+\right)}{\Psi^-}\right].
\end{aligned}
$$

And then solving forward we arrive at:

$$
\chi_0^- = \Psi^- \left(\overline{r}^f - r^m\right) + \left(1-\Psi^-\right)\left(r^w - r^m\right).
$$

which verifies that $\chi_0^- = \chi^-$. Given that $\varkappa_n^+ = \varkappa_n^-$, $\overline{r}^f$ is the same in both calculations. This concludes the proof of Proposition 2.

B. Proofs for the Infinite-Rounds Limit

B.1 Proof of Lemma 1

Here we solve for the limit as $N \to \infty$. First, let us define a matching round size step subscript as $\Delta \equiv \frac{1}{N}$. Let us abuse in notation and given that we drop the time subscript, call τ the matching round subscript in the domain $[0,1]$. Thus, realise that by definition,

$$S^+_{n+1} - S^+_n = -\lambda(N) G\left(S^+_n, S^-_n\right) \qquad \Leftrightarrow \qquad S^+_{\tau+\Delta} - S^+_\tau = -\lambda\left(\frac{1}{\Delta}\right) G\left(S^+_\tau, S^-_\tau\right)$$
$$S^-_{n+1} - S^+_n = -\lambda(N) G\left(S^+_n, S^-_n\right) \qquad \Leftrightarrow \qquad S^-_{\tau+\Delta} - S^-_\tau = -\lambda\left(\frac{1}{\Delta}\right) G\left(S^+_\tau, S^-_\tau\right);$$

where $\tau \equiv \frac{n}{N}$ for $n \in \{0, 1, 2, \ldots, N-1\}$. Next, observe that:

$$\dot{S^+_\tau} = \lim_{\Delta \to 0}\left\{\frac{S^+_{\tau+\Delta} - S^+_\tau}{\Delta}\right\} = \lim_{\Delta \to 0}\left\{\frac{-\lambda\left(\frac{1}{\Delta}\right) G\left(S^+_\tau, S^-_\tau\right)}{\Delta}\right\} = -\left[\lim_{\Delta \to 0}\left\{\frac{\lambda\left(\frac{1}{\Delta}\right)}{\Delta}\right\}\right] G\left(S^+_\tau, S^-_\tau\right)$$
$$= -\left[\lim_{N \to \infty}\left\{N\lambda\left(N\right)\right\}\right] G\left(S^+_\tau, S^-_\tau\right) = -\bar{\lambda} G\left(S^+_\tau, S^-_\tau\right),$$

and similarly:

$$\dot{S^-_\tau} = -\bar{\lambda} G\left(S^+_\tau, S^-_\tau\right).$$

Therefore:

$$\frac{\dot{S^+_\tau}}{S^+_\tau} = \frac{-\bar{\lambda} G\left(S^+_\tau, S^-_\tau\right)}{S^+_\tau} = -\bar{\lambda} G\left(1, \theta_\tau\right) \qquad \text{and} \qquad \frac{\dot{S^-_\tau}}{S^-_\tau} = \frac{-\bar{\lambda} G\left(S^+_\tau, S^-_\tau\right)}{S^-_\tau} = -\bar{\lambda} G\left(\frac{1}{\theta_\tau}, 1\right).$$

Now, since $\theta_\tau = \frac{S^-_\tau}{S^+_\tau}$,

$$\ln\left(\theta_\tau\right) = \ln\left(\frac{S^-_\tau}{S^+_\tau}\right) = \ln\left(S^-_\tau\right) - \ln\left(S^+_\tau\right).$$

Differentiating with respect of τ, then

$$\frac{\dot{\theta_\tau}}{\theta_\tau} = \frac{\dot{S^-_\tau}}{S^-_\tau} - \frac{\dot{S^+_\tau}}{S^+_\tau} = \bar{\lambda} G\left(1, \theta_\tau\right) - \bar{\lambda} G\left(\frac{1}{\theta_\tau}, 1\right).$$

Hence,

$$\dot{\theta_\tau} = \bar{\lambda}\theta_\tau\left(G\left(1, \theta_\tau\right) - G\left(\frac{1}{\theta_\tau}, 1\right)\right).$$

Matching probabilities converge to:

$$\psi^+_\tau = \bar{\lambda} G\left(1, \theta_\tau\right) \qquad \text{and} \qquad \psi^-_\tau = \bar{\lambda} G\left(\frac{1}{\theta_\tau}, 1\right) = \frac{\psi^+_\tau}{\theta_\tau}.$$

This concludes the proof of this proposition.

B.2 Auxiliary Definitions for Proposition 3

Define the probability of matching at round n for both sides of the market:

$$f^{+}(n)=\psi_{n}^{+}\left[\prod_{m=1}^{n-1}\left(1-\psi_{m}^{+}\right)\right] \quad \text{and} \quad f^{-}(n)=\psi_{n}^{-}\left[\prod_{m=1}^{n-1}\left(1-\psi_{m}^{-}\right)\right], \quad \forall n\in\{1,2,\ldots,N\}.$$

Then, observe that:

$$f^{+}(1)=\psi_{1}^{+},\ f^{+}(2)=\psi_{2}^{+}\left(1-\psi_{1}^{+}\right),f^{+}(3)=\psi_{3}^{+}\left(1-\psi_{2}^{+}\right)\left(1-\psi_{1}^{+}\right),\ldots$$
$$f^{+}(N)=\psi_{N}^{+}\left[\prod_{m=1}^{N-1}\left(1-\psi_{m}^{+}\right)\right];$$

Thus, we can write:

$$f^{+}(n)=\psi_{n}^{+}\left(1-F^{+}(n-1)\right) \quad \text{and} \quad f^{-}(n)=\psi_{n}^{-}\left(1-F^{-}(n-1)\right), \quad \forall n\in\{1,2,\ldots,N\}.$$

When as the number of rounds tends to infinity and we transform the support from n to τ, we arrive to

$$\psi_{\tau}^{+}=\lim_{N\to\infty}\left\{\frac{f^{+}\left(\frac{n}{N}\right)}{1-F^{+}\left(\frac{n-1}{N}\right)}\right\}=\lim_{\Delta\to 0}\left\{\frac{f^{+}(\tau)}{1-F^{+}(\tau-\Delta)}\right\}=\frac{f^{+}(\tau)}{1-F^{+}(\tau)} \quad \text{and}$$
$$\psi_{\tau}^{-}=\lim_{N\to\infty}\left\{\frac{f^{-}\left(\frac{n}{N}\right)}{1-F^{-}\left(\frac{n-1}{N}\right)}\right\}=\lim_{\Delta\to 0}\left\{\frac{f^{-}(\tau)}{1-F^{-}(\tau-\Delta)}\right\}=\frac{f^{-}(\tau)}{1-F^{-}(\tau)}, \quad \forall\tau\in[0,1].$$

Therefore, the cumulative distribution function conditional there is a match satisfies the following ordinary differential equation

$$\dot{F}^{+}(\tau)=\psi_{\tau}^{+}\left(1-F^{+}(\tau)\right) \quad \text{and} \quad \dot{F}^{-}(\tau)=\psi_{\tau}^{-}\left(1-F^{-}(\tau)\right).$$

Thus, solving the differential equation we arrive to

$$F^{+}(\tau)=1-e^{-\int_{0}^{\tau}\psi_{s}^{+}ds}=1-e^{-\bar{\lambda}\int_{0}^{\tau}G(1,\theta_{s})ds} \quad \text{and} \quad F^{-}(\tau)=1-e^{-\bar{\lambda}\int_{0}^{\tau}G(1/\theta_{s},1)ds}, \quad \forall\tau\in[0,1].$$

Moreover,

$$f^{+}(\tau)=\bar{\lambda}G\left(1,\theta_{\tau}\right)e^{-\bar{\lambda}\int_{0}^{\tau}G(1,\theta_{s})ds} \quad \text{and} \quad f^{-}(\tau)=\bar{\lambda}G\left(\frac{1}{\theta_{\tau}},1\right)e^{-\bar{\lambda}\int_{0}^{\tau}G(1/\theta_{s},1)ds}, \quad \forall\tau\in[0,1].$$

Hence, by construction, the probability of having a match for the atomistic investors in surplus and deficit position will be

$$\Psi^{+}=F^{+}(1)=1-e^{-\bar{\lambda}\int_{0}^{1}G(1,\theta_{s})ds} \quad \text{and} \quad \Psi^{-}=F^{-}(1)=1-e^{-\bar{\lambda}\int_{0}^{1}G\left(\frac{1}{\theta_{s}},1\right)ds}.$$

Define the weights $\{\varkappa_{\tau}^{+},\varkappa_{\tau}^{-}\}_{\tau\in[0,1]}$ as the probability weight of having a match in the interbank round process given there is a match in the settlement stage,

$$\varkappa_{\tau}^{+}\equiv\frac{f^{+}(\tau)}{F^{+}(1)} \quad \text{and} \quad \varkappa_{\tau}^{-}\equiv\frac{f^{-}(\tau)}{F^{-}(1)}, \quad \forall\tau\in[0,1].$$

B.3 Proof of Proposition 3

Now, consider the discrete round recursion for the value of χ_n^-

$$\begin{aligned}
\chi_n^- &= \psi_{n+1}^- \left(i_{n+1}^f - r^m\right) + \left(1 - \psi_{n+1}^-\right) \chi_{n+1}^- \rightarrow \\
\chi_\tau^- &= \psi_{\tau+\Delta}^- \left(i_{\tau+\Delta}^f - r^m\right) + \left(1 - \psi_{\tau+\Delta}^-\right) \chi_{\tau+\Delta}^- \rightarrow \\
\chi_{\tau+\Delta}^- - \chi_\tau^- &= -\psi_{\tau+\Delta}^- \left(i_{\tau+\Delta}^f - r^m\right) + \psi_{\tau+\Delta}^- \chi_{\tau+\Delta}^- .
\end{aligned}$$

Divide both sides by Δ, and take the limit $\Delta \rightarrow 0$, to obtain the law of motion for χ_τ^-

$$\dot{\chi}_\tau^- = \psi_\tau^- \chi_\tau^- - \psi_\tau^- \left(i_\tau^f - r^m\right). \tag{20}$$

Similarly, we follow the same operations to obtain the law of motion for χ_τ^+ :

$$\dot{\chi}_\tau^+ = \psi_\tau^+ \chi_\tau^+ - \psi_\tau^+ \left(i_\tau^f - r^m\right). \tag{21}$$

The terminal conditions in both cases are $\chi_\tau^- = \left(i_\tau^{dw} - r^m\right)$ and $\chi_\tau^+ = 0$. Acknowledge that the average interbank interest rate in continuous time can be computed as

$$r_\tau^f = r^m + (1-\eta)\chi_\tau^- + \eta\chi_\tau^+.$$

Thus, substituting we get

$$\dot{\chi}_\tau^+ = -(1-\eta)\psi_\tau^+ \left(\chi_\tau^- - \chi_\tau^+\right) \qquad \text{and} \qquad \dot{\chi}_\tau^- = \eta\psi_\tau^- \left(\chi_\tau^- - \chi_\tau^+\right).$$

Doing a change of variable, $z_\tau = \chi_\tau^- - \chi_\tau^+$. Thus,

$$\dot{z}_\tau = \left[\eta\psi_\tau^- + (1-\eta)\psi_\tau^+\right] z_\tau.$$

Define,

$$\Psi_\tau^+ \equiv \int_\tau^1 \psi_s^+ ds \qquad \text{and} \qquad \Psi_\tau^- \equiv \int_\tau^1 \psi_s^- ds.$$

Using the boundary condition $z_1 = r^w - r^m$

$$\ln\left(\frac{z_\tau}{z_1}\right) = -\int_\tau^1 \eta\psi_s^- + (1-\eta)\psi_s^+ ds = -\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right].$$

Therefore,

$$\chi_\tau^- - \chi_\tau^+ = \left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]}$$

Substituting the solution to z_τ into (20) and (21) we obtain:

$$\dot{\chi}_\tau^+ = -(1-\eta)\psi_\tau^+ \left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} \qquad \text{and} \qquad \dot{\chi}_\tau^- = \eta\psi_\tau^- \left(r^w - r^m\right) e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]}.$$

Define the residual bargained probability of match as:

$$\mathbb{P}_\tau^+ \equiv \int_\tau^1 \psi_s^+ e^{-\left[\eta\Psi_s^- + (1-\eta)\Psi_s^+\right]} ds \qquad \text{and} \qquad \mathbb{P}_\tau^- \equiv \int_\tau^1 \psi_s^- e^{-\left[\eta\Psi_s^- + (1-\eta)\Psi_s^+\right]} ds. \tag{22}$$

Hence, solving the ordinary differential equations and applying the boundary conditions we get

$$\chi_\tau^+ = (1-\eta)\left(r^w - r^m\right)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(1 - \eta\mathbb{P}_\tau^-\right). \tag{23}$$

This, in fact, is the closed form solution presented in Proposition 3.

B.4 Consistency of Solution

The final step of the proof is to verify that for the continuous-time limit, it also holds that , χ_0^- is indeed χ^-. For that, let:

$$\left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- = \left(1 - F^-(\tau)\right)\psi_\tau^-\chi_\tau^- - \left(1 - F^-(\tau)\right)\psi_\tau^-\left(r_\tau^f - r^m\right),$$

where $F^-(\tau)$ is cdf of the time distribution of matches. Rearranging terms yields:

$$\Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right) = \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right].$$

Integrating both sides over the rounds support,

$$\int_0^1 \Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right)d\tau = \int_0^1 \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right]d\tau.$$

The left-hand side yields,

$$\int_0^1 \Psi^-\varkappa_\tau^-\left(r_\tau^f - r^m\right)d\tau = \Psi^-\left[\int_0^1 \varkappa_\tau^-\left(r_\tau^f - r^m\right)dt\right] = \Psi^-\left(\overline{r}^f - r^m\right).$$

The right-hand side yields

$$\begin{aligned}
\int_0^1 \left(1 - F^-(\tau)\right)\left[\psi_\tau^-\chi_\tau^- - \dot{\chi}_\tau^-\right]d\tau &= \int_0^1 \left(1 - F^-(\tau)\right)\psi_\tau^-\chi_\tau^- - \left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- d\tau \\
&= \int_0^1 f^-(\tau)\chi_\tau^- - \left(1 - F^-(\tau)\right)\dot{\chi}_\tau^- d\tau \\
&= -\int_0^1 \frac{d}{d\tau}\left(\left(1 - F^-(\tau)\right)\chi_\tau^-\right)d\tau \\
&= -\left[\left(1 - F^-(\tau)\right)\chi_\tau^-\right]_0^1 \\
&= \left(1 - F^-(0)\right)\chi_0^- - \left(1 - F^-(1)\right)\chi_1^- \\
&= \chi_0^- - \left(1 - \Psi^-\right)\left(r^w - r^m\right).
\end{aligned}$$

Finally, joining these two expressions, the average interbank interest rate is:

$$\overline{r}^f = r^m + \left(\frac{\chi_0^- - \left(1 - \Psi^-\right)\left(r^w - r^m\right)}{\Psi^-}\right).$$

This verifies that χ_0^- is indeed χ^-. Similar steps prove that χ_0^+ is indeed χ^+.

B.5 Proof of Proposition 1

Assume a Leontief matching function so that $G(a, b) = \min\{a, b\}$. By Proposition 1, you obtain:

$$\frac{\dot{\theta}_\tau}{\theta_\tau} = \psi_\tau^+ - \psi_\tau^- = \psi_\tau^+ - \frac{\psi_\tau^+}{\theta_\tau} = \left(\frac{\theta_\tau - 1}{\theta_\tau}\right)\psi_\tau^+.$$

Thus,

$$\dot{\theta}_\tau = (\theta_\tau - 1)\,\psi_\tau^+ = (\theta_\tau - 1)\,\bar{\lambda}G(1, \theta_\tau) = (\theta_\tau - 1)\,\bar{\lambda}\min\{1, \theta_\tau\}.$$

Thus, we have that:

$$\begin{aligned}
\theta_0 > 1 \quad &\Rightarrow \quad \theta_\tau > 1 \qquad \forall\tau \in [0, 1], \\
\theta_0 = 1 \quad &\Rightarrow \quad \theta_\tau = 1 \qquad \forall\tau \in [0, 1], \\
\theta_0 < 1 \quad &\Rightarrow \quad \theta_\tau < 1 \qquad \forall\tau \in [0, 1].
\end{aligned}$$

There are three possible cases that determine the solutions to the ODE's (20) and (21):

Case $\theta_0 = 1$. In this case, we have that

$$\dot{\theta}_\tau = 0 \quad \Rightarrow \quad \theta_\tau = 1, \qquad \forall\tau \in [0, 1].$$

Thus,

$$\psi_\tau^+ = \bar{\lambda} \quad \text{and} \quad \psi_\tau^- = \bar{\lambda}, \qquad \forall\tau \in [0, 1].$$

Also,

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t) \quad \text{and} \quad \Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t); \qquad \forall t \in [0, 1].$$

Then:

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \bar{\lambda} e^{-\bar{\lambda}(1-\tau)} d\tau = \left[e^{-\bar{\lambda}(1-\tau)}\right]_t^1 = 1 - e^{-\bar{\lambda}(1-t)} \text{ and} \\
\mathbb{P}_t^- &= \int_t^1 \bar{\lambda} e^{-\bar{\lambda}(1-\tau)} d\tau = \left[e^{-\bar{\lambda}(1-\tau)}\right]_t^1 = 1 - e^{-\bar{\lambda}(1-t)}.
\end{aligned}$$

Therefore, the solution to the average payments are:

$$\chi_\tau^+ = (1-\eta)\,(r^w - r^m)\left(1 - e^{-\bar{\lambda}(1-\tau)}\right) \qquad \text{and} \qquad \chi_\tau^- = (r^w - r^m)\left(1 - \eta\left(1 - e^{-\bar{\lambda}(1-\tau)}\right)\right)$$

This allows us to arrive to the interbank-reserves interest rate spread:

$$r_\tau^f - r^m = (1-\eta)\,(r^w - r^m).$$

This implies that $\bar{r}^f = r^m + (1-\eta)(r^w - r^m)$ as in the statement of the Proposition. For $\eta \to 0$, we arrive to the intuitive result of $\bar{r}^f = r^w$ and for $\eta \to 1$, $\bar{r}^f = r^m$.

Case $\theta_0 > 1$. In this case, we have that $\theta_\tau > 1$ for every $\tau \in [0,1]$. Thus, it follows that the law of motion for tightness is

$$\theta_\tau = \bar{\lambda}(\theta_\tau - 1) \quad \Rightarrow \quad \theta_\tau = 1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}, \qquad \forall \tau \in [0,1].$$

$$\psi_\tau^+ = \bar{\lambda} \quad \text{and} \quad \psi_\tau^- = \frac{\bar{\lambda}}{1 + (\theta_0 - 1)e^{\bar{\lambda}\tau}}, \qquad \forall \tau \in [0,1].$$

Applying this results:

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t)$$

$$\Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \frac{\bar{\lambda}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}} d\tau = \left[\bar{\lambda}\tau - \ln\left(1+(\theta_0-1)e^{\bar{\lambda}\tau}\right)\right]_t^1 = \bar{\lambda}(1-t) - \ln\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}t}}\right).$$

Following,

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \psi_\tau^+ e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \bar{\lambda} e^{-\bar{\lambda}(1-\tau) + \eta \ln\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)^\eta \bar{\lambda} e^{-\bar{\lambda}(1-\tau)} d\tau = \left[\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}\tau}}{(1-\eta)(\theta_0-1)e^{\bar{\lambda}}}\right)\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)^\eta\right]_t^1 \\
&= \left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{(1-\eta)(\theta_0-1)e^{\bar{\lambda}}}\right) - \left(\frac{1+(\theta_0-1)e^{\bar{\lambda}t}}{(1-\eta)(\theta_0-1)e^{\bar{\lambda}}}\right)\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}t}}\right)^\eta \\
&= \frac{\theta_1\left(1-\left(\frac{\theta_\tau}{\theta_1}\right)^{1-\eta}\right)}{(1-\eta)(\theta_1-1)} = \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_1 - \theta_\tau^{1-\eta}\theta_1^\eta}{\theta_1 - 1}\right) = \left(\frac{1}{1-\eta}\right)\left(\frac{\bar{\theta}}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar{\theta}^{1-\eta} - \theta}{\bar{\theta}-1}\right).
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}_t^- &= \int_t^1 \psi_\tau^- e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \left(\frac{\bar{\lambda}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right) e^{-\bar{\lambda}(1-\tau) + \eta\ln\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)^\eta \left(\frac{\bar{\lambda}e^{-\bar{\lambda}(1-\tau)}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right) d\tau = \left[\left(\frac{-1}{\eta(\theta_0-1)e^{\bar{\lambda}}}\right)\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}\right)^\eta\right]_t^1 \\
&= \left(\frac{1}{\eta(\theta_0-1)e^{\bar{\lambda}}}\right)\left(\frac{1+(\theta_0-1)e^{\bar{\lambda}}}{1+(\theta_0-1)e^{\bar{\lambda}t}}\right)^\eta - \left(\frac{1}{\eta(\theta_0-1)e^{\bar{\lambda}}}\right) \\
&= \frac{\left(\frac{\theta_1}{\theta_\tau}\right)^\eta - 1}{\eta(\theta_1-1)}
\end{aligned}$$

Therefore, using (23), we have that:

$$\chi_\tau^+ = (1-\eta)(r^w - r^m)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = (r^w - r^m)\left(1 - \eta\mathbb{P}_\tau^-\right),$$

which implies:

$$\chi_\tau^+ = (r^w - r^m)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta \theta_1^{1-\eta} - \theta_\tau}{\theta_1 - 1}\right) \qquad \text{and} \qquad \chi_\tau^- = (r^w - r^m)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta \theta_1^{1-\eta} - 1}{\theta_1 - 1}\right).$$

From here, substitute for $\tau = 0$ and obtain $\{\chi_0^+, \chi_0^-\}$.

Recall that the interbank rates satisfy:

$$r_n^f \equiv r^m + (1-\eta)\,\chi_{n+1}^- + \eta\chi_{n+1}^+.$$

Now from here we obtain the interbank-reserves interest rate spread,

$$\begin{aligned} r_\tau^f - r^m &= \\ &= (r^w - r^m)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta \theta_1^{1-\eta} - 1}{\theta_1 - 1} - \eta\left(\frac{\theta_\tau - 1}{\theta_1 - 1}\right)\right). \end{aligned}$$

The matching probabilities are:

$$\psi_\tau^+ = 1 - e^{-\bar{\lambda}} \qquad \text{and} \qquad \psi^- = 1 - e^{-\bar{\lambda}\int_0^1 \frac{1}{\theta_\tau}d\tau} = 1 - e^{-\bar{\lambda}\int_0^1 \frac{1}{1+(\theta_0-1)e^{\bar{\lambda}\tau}}d\tau} = 1 - \left(\frac{(\theta_0-1)e^{\bar{\lambda}}+1}{\theta_0}\right)e^{-\bar{\lambda}} = \frac{1-e^{-\bar{\lambda}}}{\theta_0}.$$

Finally, let us compute the average interbank interest rate

$$\begin{aligned} \bar{r}^f &= r^m + \left(\frac{\chi_0^- - (1-\psi^-)(r^w - r^m)}{\psi^-}\right) \\ &= r^m + \left(\frac{\chi_0^-}{\psi^-}\right) - \left(\frac{1-\psi^-}{\psi^-}\right)(r^w - r^m) \\ &= r^m + \left(\frac{1}{\psi^-}\right)\left(\frac{\theta_1}{\theta_0}\right)^\eta \left(\frac{\theta_0^\eta \theta_1^{1-\eta} - 1}{\theta_1 - 1}\right)(r^w - r^m) + \left(1 - \left(\frac{1}{\psi^-}\right)\right)(r^w - r^m) \\ &= r^w + \left(\frac{r^w - r^m}{\psi^-}\right)\left[\left(\frac{\theta_1}{\theta_0}\right)^\eta \left(\frac{\theta_0^\eta \theta_1^{1-\eta} - 1}{\theta_1 - 1}\right) - 1\right] \\ &= r^w - \left(\frac{r^w - r^m}{\psi^-}\right)\left(\frac{\left(\frac{\theta_1}{\theta_0}\right)^\eta - 1}{\theta_1 - 1}\right) \\ &= r^w - (r^w - r^m)\left(\frac{\theta_0}{\theta_0 - 1}\right)\left(\left(\frac{\theta_1}{\theta_0}\right)^\eta - 1\right)\left(\frac{1}{e^{\bar{\lambda}} - 1}\right). \end{aligned}$$

Which is the formula in the expression.Notice that, if $\eta \to 0$, we arrive to the intuitive result of $\bar{r}^f = r^w$ and $\eta \to 1$, substitute for θ_1 in terms of θ_0 and this shows that at this limit $\bar{r}^f = r^m$.

Case $\theta_0 < 1$. In this case, we have that $\theta_\tau < 1$ for every $\tau \in [0,1]$ and thus:

$$\theta_\tau = \bar{\lambda}\,(\theta_\tau - 1)\,\theta_\tau \quad \Rightarrow \quad \theta_\tau = \frac{1}{1 + \left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}, \qquad \forall \tau \in [0,1].$$

$$\psi_\tau^+ = \frac{\bar{\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}t}} \quad \text{and} \quad \psi_\tau^- = \bar{\lambda}, \qquad \forall \tau \in [0,1].$$

Then,

$$\Psi_t^+ = \int_t^1 \psi_\tau^+ d\tau = \int_t^1 \frac{\bar{\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}} d\tau = \left[\bar{\lambda}\tau - \ln\left(1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}\right)\right]_t^1 = \bar{\lambda}(1-t) - \ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}t}}\right)$$

$$\Psi_t^- = \int_t^1 \psi_\tau^- d\tau = \int_t^1 \bar{\lambda} d\tau = \bar{\lambda}(1-t).$$

Following,

$$\begin{aligned}
\mathbb{P}_t^+ &= \int_t^1 \psi_\tau^+ e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \left(\frac{\bar{\lambda}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right) e^{-\bar{\lambda}(1-\tau)+(1-\eta)\ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right)^{1-\eta} \left(\frac{\bar{\lambda}e^{-\bar{\lambda}(1-\tau)}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right) d\tau = \left[\left(\frac{-1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right)^{1-\eta}\right]_t^1 \\
&= \left(\frac{1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}t}}\right)^{1-\eta} - \left(\frac{1}{(1-\eta)\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right) \\
&= \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_1}{1-\theta_1}\right)\left(\left(\frac{\theta_t}{\theta_1}\right)^{1-\eta} - 1\right) \\
&= \left(\frac{1}{1-\eta}\right)\left(\frac{\theta_t^{1-\eta}\theta_1^{\eta} - \theta_1}{1-\theta_1}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}_t^- &= \int_t^1 \psi_\tau^- e^{-\left[\eta\Psi_\tau^- + (1-\eta)\Psi_\tau^+\right]} d\tau = \int_t^1 \bar{\lambda} e^{-\bar{\lambda}(1-\tau)+(1-\eta)\ln\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right)} d\tau \\
&= \int_t^1 \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}\right)^{1-\eta} \bar{\lambda}e^{-\bar{\lambda}(1-\tau)} d\tau \\
&= \left[\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)^{\eta}\right]_t^1 \\
&= \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right) - \left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}{\eta\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)\left(\frac{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}t}}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}}\right)^{\eta} \\
&= \left(\frac{1}{\eta}\right)\left(\frac{1-\left(\frac{\theta_1}{\theta_t}\right)^{\eta}}{1-\theta_1}\right).
\end{aligned}$$

Therefore,

$$\chi_\tau^+ = (1-\eta)\left(r^w - r^m\right)\mathbb{P}_\tau^+ \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(1-\eta\mathbb{P}_\tau^-\right).$$

Thus:

$$\chi_\tau^+ = \left(r^w - r^m\right)\left(\frac{\theta_\tau^{1-\eta}\theta_1^\eta - \theta_1}{1-\theta_1}\right) = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - \theta_\tau}{\theta_1 - 1}\right).$$

and

$$\chi_\tau^- = \left(r^w - r^m\right)\left(1 - \left(\frac{1-\left(\frac{\theta_1}{\theta_\tau}\right)^\eta}{1-\theta_1}\right)\right) = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - 1}{\theta_1 - 1}\right).$$

Hence, in summary we obtain:

$$\chi_\tau^+ = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_1^{1-\eta}\theta_\tau^\eta - \theta_\tau}{\theta_1 - 1}\right) \qquad \text{and} \qquad \chi_\tau^- = \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta\theta_1^{1-\eta} - 1}{\theta_1 - 1}\right).$$

If we substitute for $t = 0$, we arrive to the expressions for $\left\{\chi_0^+, \chi_0^-\right\}$ employed in the proposition.

Recall that the interbank rates satisfy:

$$r_n^f \equiv r^m + (1-\eta)\,\chi_{n+1}^- + \eta\chi_{n+1}^+.$$

Now from here we obtain the interbank-reserves interest rate spread,

$$\begin{aligned} r_\tau^f - r^m &= \\ &= \left(r^w - r^m\right)\left(\frac{\theta_1}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta\theta_1^{1-\eta} - 1}{\theta_1 - 1} - \eta\left(\frac{\theta_\tau - 1}{\theta_1 - 1}\right)\right). \end{aligned}$$

Now the probability of finding a match during the interbank matching process can be computed as,

$$\psi_\tau^+ = 1 - e^{-\bar{\lambda}\int_0^1 \theta_\tau d\tau} = 1 - e^{-\bar{\lambda}\int_0^1 \frac{1}{1+\left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}\tau}} d\tau} = 1 - \theta_0 e^{-\bar{\lambda}}\left(1 + \left(\frac{1-\theta_0}{\theta_0}\right)e^{\bar{\lambda}}\right) = \theta_0\left(1 - e^{-\bar{\lambda}}\right)$$

and

$$\psi^- = 1 - e^{-\bar{\lambda}}.$$

From here, we compute the average interbank interest rate:

$$\begin{aligned}
\overline{r}^f &= r^m + \left(\frac{\chi_0^- - (1-\psi^-)(r^w - r^m)}{\psi^-}\right) \\
&= r^m + \left(\frac{\chi_0^-}{\psi^-}\right) - \left(\frac{1-\psi^-}{\psi^-}\right)(r^w - r^m) \\
&= r^m + \left(\frac{1}{\psi^-}\right)\left(\frac{\theta_1}{\theta_0}\right)^{\eta}\left(\frac{1-\theta_0^{\eta}\theta_1^{1-\eta}}{1-\theta_1}\right)(r^w - r^m) + \left(1 - \left(\frac{1}{\psi^-}\right)\right)(r^w - r^m) \\
&= r^w + \left(\frac{r^w - r^m}{\psi^-}\right)\left[\left(\frac{\theta_1}{\theta_0}\right)^{\eta}\left(\frac{1-\theta_0^{\eta}\theta_1^{1-\eta}}{1-\theta_1}\right) - 1\right] \\
&= r^w - \left(\frac{r^w - r^m}{\psi^-}\right)\left(\frac{1 - \left(\frac{\theta_1}{\theta_0}\right)^{\eta}}{1-\theta_1}\right) \\
&= r^w - \left(1 - \left(\frac{\theta_1}{\theta_0}\right)^{\eta}\right)\left(\left(\frac{\theta_0}{\theta_0 - 1}\right) + e^{\bar{\lambda}}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}} - 1}\right) \\
&= r^w - \left(1 - \left(\frac{\theta_1}{\theta_0}\right)^{\eta}\right)\left(\left(\frac{\theta_0}{1 - \theta_0}\right) + e^{\bar{\lambda}}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}} - 1}\right) \\
&= r^w - \left(1 - \left(\frac{\theta_1}{\theta_0}\right)^{\eta}\right)\left(\left(\frac{\theta_0 + e^{\bar{\lambda}}(1-\theta_0)}{1 - \theta_0}\right)\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}} - 1}\right).
\end{aligned}$$

Notice that, if $\eta \to 0$, we arrive to the intuitive result of $\overline{r}^f = r^w$. Similarly, we obtain that $\eta \to 1$, leads to $\overline{r}^f = r^m$.

Then, we have the relationship:

$$(\theta_1)^{-1}\theta_0 = \theta_0 + (1-\theta_0)e^{\bar{\lambda}}$$

in which case:

$$\begin{aligned}
\overline{r}^f &= r^w - \left(1 - \left(\frac{\theta_1}{\theta_0}\right)^{\eta}\right)\left(\left(\frac{\theta_0 + \left((\theta_1)^{-1} - 1\right)\theta_0}{1-\theta_0}\right)\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}} - 1}\right) \\
&= r^w - \left(\left(\frac{\theta_1}{\theta_0}\right)^{\eta} - 1\right)\left(\frac{1}{\theta_1}\frac{\theta_0}{\theta_0 - 1}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}} - 1}\right).
\end{aligned}$$

Alternative representation. The statement in the proposition uses the following alternative representation.

$$\chi_\tau^+ = (r^w - r^m)\left(\frac{\theta_1}{\theta_\tau}\right)^{\eta}\left(\frac{\theta_\tau^{\eta}\theta_1^{1-\eta} - \theta_\tau}{\theta_1 - 1}\right) = (r^w - r^m)\left(\frac{\theta_1 - \theta_1^{\eta}\theta_\tau^{1-\eta}}{\theta_1 - 1}\right)$$

and

$$\chi_\tau^- = (r^w - r^m)\left(\frac{\theta_1}{\theta_\tau}\right)^{\eta}\left(\frac{\theta_\tau^{\eta}\theta_1^{1-\eta} - 1}{\theta_1 - 1}\right) = (r^w - r^m)\left(\frac{\theta_1 - \theta_1^{\eta}\theta_\tau^{-\eta}}{\theta_1 - 1}\right).$$

Then, using that:

$$r_n^f - r^m \equiv (1-\eta)\chi_{n+1}^- + \eta\chi_{n+1}^+ = (r^w - r^m)\left(\frac{\theta_1 - \eta\theta_1^{\eta}\theta_\tau^{1-\eta} - (1-\eta)\theta_1^{\eta}\theta_\tau^{-\eta}}{\theta_1 - 1}\right).$$

Thus, letting ϕ_τ denote the endogenous bargaining power of the corresponding round, we obtain:

$$\phi_\tau = 1 - \left(\frac{\theta_1 - \eta\theta_1^\eta \theta_\tau^{1-\eta} - (1-\eta)\, \theta_1^\eta \theta_\tau^{-\eta}}{\theta_1 - 1} \right) = \left(\frac{\left(\frac{\theta_1}{\theta_\tau}\right)^\eta (\eta\theta_\tau + (1-\eta)) - 1}{\theta_1 - 1} \right).$$

B.6 Proof of Corollary 4

The average interbank rate is:

$$\bar{R}^f = R^m + \frac{\chi^+}{\Psi^+}.$$

Thus, we obtain that:

$$\begin{aligned} \bar{R}^f &= R^m + \frac{(R^w - R^m)\left(\frac{\bar{\theta}}{\theta}\right)^\eta \left(\frac{\theta^\eta \bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1}\right)}{\Psi^+} \\ &= (1 - \phi(\theta))\, R^w + \phi(\theta)\, R^m, \end{aligned}$$

where

$$\phi(\theta) = 1 - \frac{\left(\frac{\bar{\theta}}{\theta}\right)^\eta \left(\frac{\theta^\eta \bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1}\right)}{\Psi^+} = 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\left(\bar{\theta} - 1\right)\Psi^+}.$$

Recall that

$$\Psi^+ = \begin{cases} 1 - e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1 - e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases},$$

and that:

$$\bar{\theta} = \begin{cases} 1 + (\theta - 1)\exp\left(\bar{\lambda}\right) & \text{if } \theta \geq 1 \\ \left(1 + \left(\theta^{-1} - 1\right)\exp\left(\bar{\lambda}\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

Thus, we have the following two cases:

- If $\theta > 1$ we have that:

$$\begin{aligned} 1 - \phi(\theta) &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{(\theta - 1)\exp\left(\bar{\lambda}\right)\left(1 - \exp\left(-\bar{\lambda}\right)\right)} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{(\theta - 1)\exp\left(\bar{\lambda}\right) + 1 - \theta} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{1 + (\theta - 1)\exp\left(\bar{\lambda}\right) - \theta} \\ &= 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\bar{\theta} - \theta}. \end{aligned}$$

  Hence:

$$\phi(\theta) = \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^\eta}{\bar{\theta} - \theta}.$$

- This number is between zero and one since $\bar{\theta} > \theta \to \bar{\theta} > \theta^{1-\eta}\bar{\theta}^\eta > \theta$.

- If $\theta < 1$ we have that:

$$\begin{aligned}
1-\phi(\theta) &= 1-\frac{\bar{\theta}-\theta^{1-\eta}\bar{\theta}^{\eta}}{\left(\bar{\theta}-1\right)\theta\left(1-e^{-\bar{\lambda}}\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\bar{\theta}^{\eta-1}}{\left(1-\bar{\theta}^{-1}\right)\left(1-e^{-\bar{\lambda}}\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\left(1+\left(\theta^{-1}-1\right)\exp\left(\bar{\lambda}\right)\right)^{(1-\eta)}}{\left(\theta^{-1}-1\right)\left(\exp\left(\bar{\lambda}\right)-1\right)} \\
&= 1-\frac{\theta^{-1}-\theta^{-\eta}\left(\bar{\theta}^{-1}\right)^{(1-\eta)}}{\bar{\theta}^{-1}-\theta^{-1}}.
\end{aligned}$$

Hence, we have:

$$\phi(\theta) = 1-\frac{\bar{\theta}^{-1}-\theta^{-\eta}\left(\bar{\theta}^{-1}\right)^{(1-\eta)}}{\bar{\theta}^{-1}-\theta^{-1}}.$$

This shows the symmetry property. The bargaining power falls between zero and one since $\bar{\theta}^{-1} > \theta^{-1} \rightarrow \bar{\theta}^{-1} > \theta^{-(1-\eta)}\bar{\theta}^{-\eta} > \theta^{-1}$.

B.7 Symmetry

So far, we have show that given θ, the after-trade tightness is given by:

$$\bar{\theta} = \theta_1 = \begin{cases} 1+(\theta-1)\exp\left(\bar{\lambda}\right) & \text{if } \theta > 1 \\ 1 & \text{if } \theta = 1 \\ \left(1+\left(\theta^{-1}-1\right)\exp\left(\bar{\lambda}\right)\right)^{-1} & \text{if } \theta < 1 \end{cases}.$$

Trading probabilities are given by:

$$\Psi^{+} = \begin{cases} 1-e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1-e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases}, \qquad \Psi^{-} = \begin{cases} \left(1-e^{-\bar{\lambda}}\right)\theta^{-1} & \text{if } \theta > 1 \\ 1-e^{-\bar{\lambda}} & \text{if } \theta \leq 1 \end{cases}.$$

Thus, the slopes of the liquidity-yield function are given by:

$$\chi^{+} = \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right) \text{ and } \chi^{-} = \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right).$$

Next, we simplify the solution.

Case $\theta < 1$. In this case:

$$\bar{\theta} = \left(1+\left(\theta^{-1}-1\right)\exp\left(\lambda\right)\right)^{-1}.$$

The following calculations are useful. Observe that,

$$\frac{\bar{\theta}}{\theta} = \frac{\theta}{\theta\left(\theta+(1-\theta)\exp\left(\lambda\right)\right)} = \frac{1}{\theta+(1-\theta)\exp\left(\lambda\right)}$$

And from here, that,

$$1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}=1-\left(\frac{1}{\theta+(1-\theta)\exp(\lambda)}\right)^{\eta}$$

and,

$$1-\bar{\theta}=1-\frac{1}{(1+(\theta^{-1}-1)\exp(\lambda))}=\frac{(1-\theta)\exp(\lambda)}{\theta+(1-\theta)\exp(\lambda)}.$$

Thus, the "Cobb-Douglas" term, satisfies:

$$\theta^{\eta}\bar{\theta}^{1-\eta}=\frac{\theta^{\eta}\left(1+\left(\theta^{-1}-1\right)\exp(\lambda)\right)^{\eta}}{(1+(\theta^{-1}-1)\exp(\lambda))}=\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(1+(\theta^{-1}-1)\exp(\lambda))}=\theta\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(\theta+(1-\theta)\exp(\lambda))}.$$

Now, define:

$$h^{+}\equiv\theta-\theta^{\eta}\bar{\theta}^{1-\eta}=\theta\left(1-\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(\theta+(1-\theta)\exp(\lambda))}\right)$$

and

$$h^{-}\equiv 1-\theta^{\eta}\bar{\theta}^{1-\eta}=\left(1-\theta\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(\theta+(1-\theta)\exp(\lambda))}\right).$$

With these definitions, we obtain the following expression for the slopes of the liquidity yield:

$$\chi^{+}=(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}h^{+}\left(\frac{1}{1-\bar{\theta}}\right)=(r^{w}-r^{m})\left(\frac{1}{\theta+(1-\theta)\exp(\lambda)}\right)^{\eta}\frac{\theta\left(1-\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(\theta+(1-\theta)\exp(\lambda))}\right)}{\frac{(1-\theta)\exp(\lambda)}{\theta+(1-\theta)\exp(\lambda)}}$$

and

$$\chi^{-}=(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}h^{-}\left(\frac{1}{1-\bar{\theta}}\right)=(r^{w}-r^{m})\left(\frac{1}{\theta+(1-\theta)\exp(\lambda)}\right)^{\eta}\frac{\left(1-\theta\frac{(\theta+(1-\theta)\exp(\lambda))^{\eta}}{(\theta+(1-\theta)\exp(\lambda))}\right)}{\frac{(1-\theta)\exp(\lambda)}{\theta+(1-\theta)\exp(\lambda)}}.$$

Define:

$$\rho\equiv(1-\theta)\exp(\lambda)$$

Then

$$\begin{aligned}\chi^{+}&=(r^{w}-r^{m})\left(\frac{1}{\theta+\rho}\right)^{\eta}\frac{\theta\left(1-\frac{(\theta+\rho)^{\eta}}{\theta+\rho}\right)}{\frac{\rho}{\theta+\rho}}=(r^{w}-r^{m})\frac{\theta}{\rho}(\theta+\rho)^{-\eta}((\theta+\rho)-(\theta+\rho)^{\eta})\\&=(r^{w}-r^{m})\,\theta\frac{\left((\theta+(1-\theta)\exp(\lambda))^{1-\eta}-1\right)}{(1-\theta)\exp(\lambda)},\end{aligned}$$

and

$$\chi^{-} = (r^w - r^m)\left(\frac{1}{\theta+\rho}\right)^{\eta} \frac{\left(1 - \frac{(\theta+\rho)^{\eta}}{\theta+\rho}\right)}{\frac{\rho}{\theta+\rho}} = (r^w - r^m)\left(\frac{1}{\theta+\rho}\right)^{\eta} \frac{1}{\rho}(\theta+\rho)^{-\eta}\left((\theta+\rho) - \theta(\theta+\rho)^{\eta}\right)$$

$$= (r^w - r^m)\frac{\left((\theta + (1-\theta)\exp(\lambda))^{1-\eta} - \theta\right)}{(1-\theta)\exp(\lambda)}.$$

Therefore, in summary:

$$\chi^{-} = (r^w - r^m)\frac{(\theta + (1-\theta)\exp(\lambda))^{1-\eta} - \theta}{(1-\theta)\exp(\lambda)},$$

and

$$\chi^{+} = (r^w - r^m)\frac{\theta(\theta + (1-\theta)\exp(\lambda))^{1-\eta} - \theta}{(1-\theta)\exp(\lambda)}.$$

The resulting average OTC market rate is determined by the average of Nash bargaining over the positions and is given by:

$$\overline{r}^f(\theta, r^m, r^w) \equiv r^m + (r^w - r^m)\frac{(\theta + (1-\theta)\exp(\lambda))^{1-\eta} - 1}{1 - \exp(\lambda)} \tag{24}$$

Case $\theta > 1$. In this case:

$$\bar{\theta} = 1 + (\theta - 1)\exp(\bar{\lambda}).$$

The following calculations are useful. Observe that,

$$\frac{\bar{\theta}}{\theta} = \frac{1 + (\theta-1)\exp(\bar{\lambda})}{\theta} = \theta^{-1} + \left(1 - \theta^{-1}\right)\exp(\bar{\lambda}).$$

And from here, that,

$$1 - \left(\frac{\bar{\theta}}{\theta}\right)^{\eta} = 1 - \left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp(\bar{\lambda})\right)^{\eta}$$

and,

$$1 - \bar{\theta} = 1 - \theta^{-1} - \left(1 - \theta^{-1}\right)\exp(\bar{\lambda}) = \left(1 - \theta^{-1}\right)(1 - \exp(\lambda)).$$

Thus, :

$$\theta^{\eta}\bar{\theta}^{1-\eta} = \theta\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp(\bar{\lambda})\right)^{1-\eta}.$$

Now, define:

$$h^{+} \equiv \theta - \theta^{\eta}\bar{\theta}^{1-\eta} = \theta\left(1 - \left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp(\bar{\lambda})\right)^{1-\eta}\right)$$

and

$$h^{-} \equiv 1 - \theta^{\eta}\bar{\theta}^{1-\eta} = \left(1 - \theta\left(\theta^{-1} + \left(1-\theta^{-1}\right)\exp(\bar{\lambda})\right)^{1-\eta}\right).$$

With these definitions, we obtain the following expression for the slopes of the liquidity yield:

$$\begin{aligned}\chi^{+} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\frac{h^{+}}{1-\bar{\theta}}\\ &= \left(r^{w}-r^{m}\right)\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{\eta}\frac{\left(\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta}-1\right)}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}\\ &= \frac{\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)-\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}.\end{aligned}$$

Likewise, we obtain that:

$$\begin{aligned}\chi^{-} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\frac{h^{-}}{1-\bar{\theta}}\\ &= \left(r^{w}-r^{m}\right)\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{\eta}\frac{\left(\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)\right)^{1-\eta}-\theta^{-1}\right)}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}\\ &= \frac{\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)-\theta^{-1}\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}.\end{aligned}$$

Next, we prove the symmetry:

$$\chi^{-}\left(\theta,\eta\right)=\Delta-\chi^{+}\left(\theta^{-1},1-\eta\right)$$

Using the previous formulas:

$$\begin{aligned}\chi^{-}\left(\theta,\eta\right) &= \left(r^{w}-r^{m}\right)\left(1-\frac{\theta^{-1}\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}-\theta^{-1}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right)\\ &= \left(r^{w}-r^{m}\right)\frac{\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)-\theta^{-1}\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}.\end{aligned}$$

Likewise, we have that:

$$\chi^{+}\left(\theta,\eta\right)=\Delta-\chi^{-}\left(\theta^{-1},1-\eta\right)$$

Using the previous formulas,

$$\begin{aligned}\chi^{+}\left(\theta,\eta\right) &= \left(r^{w}-r^{m}\right)\left(1-\frac{\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}-\theta^{-1}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right)\\ &= \left(r^{w}-r^{m}\right)\left(\frac{\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)-\left(\theta^{-1}+\left(1-\theta^{-1}\right)\exp\left(\lambda\right)\right)^{\eta}}{\left(1-\theta^{-1}\right)\exp\left(\lambda\right)}\right).\end{aligned}$$

Summary. [TBA]

B.8 Volume Distribution

Fix a partiucular date t and let S be the shorter side of the market, that is $S=\min\left\{S^{-},S^{+}\right\}$. Since $\theta>1$ implies that $\theta_{\tau}>1$ at all trading sessions, we know that the shorter side of the market remains the shorter side at all trading sessions. Hence, the continuous-time limit of equation (8) yields a law of motion for the shorter side:

$$\dot{S} = -\lambda S.$$

Thus, we have that:

$$S_\tau = S_0 \exp\left[-\lambda \tau\right].$$

As a result, we know that the total volume of trade is:

$$V = S_0 - S_1 = S_0(1 - \exp\left[-\lambda \tau\right]).$$

Moreover, the transactions per instant of time are:

$$g_\tau = \lambda G\left[S_\tau^+, S_\tau^-\right] = \lambda S_\tau \min\left[\frac{S_\tau^+}{\min\left\{S^-, S^+\right\}}, \frac{S_\tau^-}{\min\left\{S^-, S^+\right\}}\right] = \lambda S_\tau.$$

Hence, the volume distribution in the interbank market is:

$$v_\tau = \frac{g_\tau}{V} = \frac{\lambda \exp\left[-\lambda \tau\right]}{1 - \exp\left(\lambda \tau\right)}.$$

The volume of discount-window loans is given by:

$$W = S^- - V = \begin{cases} S^+\left(\exp\left[-\lambda \tau\right] + (\theta - 1)\right) & \text{if } \theta > 1 \\ \\ S^- \exp\left[-\lambda \tau\right] & \text{if } \theta \leq 1. \end{cases}$$

Thus, the volume of discount loans to overall interbank loans is given by:

$$\frac{W}{V} = \frac{\exp\left[-\lambda \tau\right] + (\theta - 1)\, \mathbb{I}_{[\theta > 1]}}{1 - \exp\left[-\lambda \tau\right]}.$$

B.9 Dispersion

We can produce different metrics of dispersion in the interbank market.

- For $\theta = 1$
$$r_\tau^f = r^m + (1 - \eta)\left(r^w - r^m\right).$$
- For $\theta > 1$
$$r_\tau^f = r^m + \left(r^w - r^m\right)\left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{\theta_\tau^\eta \bar{\theta}^{1-\eta} - 1}{\bar{\theta} - 1} - \eta\left(\frac{\theta_\tau - 1}{\bar{\theta} - 1}\right)\right).$$
- For $\theta < 1$
$$r_\tau^f = r^m + \left(r^w - r^m\right)\left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{1 - \theta_\tau^\eta \bar{\theta}^{1-\eta}}{1 - \bar{\theta}} - \eta\left(\frac{1 - \theta_\tau}{1 - \bar{\theta}}\right)\right).$$

Clearly, for $\tau = 1$ we have that:

$$r_1^f = r^m + (1 - \eta)\left(r^w - r^m\right).$$

Next, we have that:

- For $\theta = 1$
$$r_1^f - r_\tau^f = 0.$$
Hence, the max-min and the standard deviation of interbank rates is constant.

- For $\theta > 0$, we have that:
$$\begin{aligned} r_1^f - r_\tau^f &= (r^w - r^m)\left[(1-\eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1-\eta) - \left(\theta_\tau^\eta \bar{\theta}^{1-\eta} - \eta\theta_\tau\right)}{\bar{\theta}-1}\right)\right] \\ &= (r^w - r^m)\left[(1-\eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1-\eta) - \theta_\tau\left(\left(\frac{\bar{\theta}}{\theta_\tau}\right)^{1-\eta} - \eta\right)}{\bar{\theta}-1}\right)\right] \end{aligned}$$

- For $\theta < 0$, we have that:
$$\begin{aligned} r_1^f - r_\tau^f &= (r^w - r^m)\left[(1-\eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1-\eta) - \left(\theta_\tau^\eta \bar{\theta}^{1-\eta} - \eta\theta_\tau\right)}{\bar{\theta}-1}\right)\right] \\ &= (r^w - r^m)\left[(1-\eta) + \left(\frac{\bar{\theta}}{\theta_\tau}\right)^\eta \left(\frac{(1-\eta) - \theta_\tau\left(\left(\frac{\bar{\theta}}{\theta_\tau}\right)^{1-\eta} - \eta\right)}{\bar{\theta}-1}\right)\right]. \end{aligned}$$

Clearly, for $\tau = 1$ we have that:

- For $\theta > 1$
$$\rho \equiv \frac{\bar{\theta}}{\theta} = \theta^{-1} + \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right) > 1.$$
The derivatives of this ratio are:
$$\rho_\theta = -\frac{\exp\left(\bar{\lambda}\right) - 1}{\theta^2} < 1$$
and
$$\rho_\lambda = \left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right) > 0.$$

- For $\theta < 1$
$$\frac{\bar{\theta}}{\theta} = \left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^{-1} < 1.$$
The derivatives of this ratio are:
$$\rho_\theta = -\frac{\exp\left(\bar{\lambda}\right) - 1}{\left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^2} < 1$$
and
$$\rho_\lambda = \frac{\left(1-\theta^{-1}\right)\exp\left(\bar{\lambda}\right)}{\left(\theta + (1-\theta)\exp\left(\bar{\lambda}\right)\right)^2} > 0.$$

- Then, we have that:

$$r_1^f - r_\tau^f \quad = \quad (r^w - r^m)\left[(1-\eta) + \rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right]$$

Then, taking total derivatives with respect to λ and θ we obtain:

$$\eta\frac{d\rho}{\rho}\left[\rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - \frac{d\theta_\tau}{\theta_\tau}\left[\rho^\eta\left(\frac{\theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - (1-\eta)\frac{d\rho}{\rho}\left[\rho^\eta\left(\frac{\theta_\tau\rho^{1-\eta}}{\bar{\theta} - 1}\right)\right] - \frac{d\left(\bar{\theta} - 1\right)}{\left(\bar{\theta} - 1\right)}\left[\rho^\eta\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right].$$

Grouping terms:

$$\left[\eta\frac{d\rho}{\rho} - \frac{d\left(\bar{\theta} - 1\right)}{\left(\bar{\theta} - 1\right)}\right]\left[\rho^\eta\left(\frac{(1-\eta) - \theta_\tau\left(\rho^{1-\eta} - \eta\right)}{\bar{\theta} - 1}\right)\right] - \frac{d\theta_\tau\left(\rho - \eta\rho^\eta\right) + d\rho(1-\eta)\theta_\tau}{\bar{\theta} - 1}.$$

For $\theta > 1$, The term $\frac{d\rho}{\rho} < 0$. In turn, $d\bar{\theta} - 1 > 0$

C. Proof of Derivatives with respect to θ

Case $\theta < 1$. Trading probabilities are given by:

$$\Psi^{+} = \begin{cases} 1 - e^{-\bar{\lambda}} & \text{if } \theta \geq 1 \\ \theta\left(1 - e^{-\bar{\lambda}}\right) & \text{if } \theta < 1 \end{cases}, \qquad \Psi^{-} = \begin{cases} \left(1 - e^{-\bar{\lambda}}\right)\theta^{-1} & \text{if } \theta > 1 \\ 1 - e^{-\bar{\lambda}} & \text{if } \theta \leq 1 \end{cases}.$$

Next, we explore the derivatives of the liquidity yield, varying the market tightness. In the special case where $\theta < 1$. We have the following:

$$\overline{R}^{f}_{\theta} \equiv (R^{w} - R^{m}) \frac{(1-\eta)}{(\theta + (1-\theta)\exp(\lambda))^{\eta}} \in [0, (1-\eta)].$$

The second derivative in turn satisfies:

$$\overline{R}^{f}_{\theta\theta} > (R^{w} - R^{m}) \frac{\eta(1-\eta)(\exp(\lambda) - 1)}{(\theta + (1-\theta)\exp(\lambda))^{1+\eta}} > 0.$$

Thus, the $\overline{R}^{f}$ is convex in θ when $\theta < 1$.

Using (5), we obtain:

$$\chi^{+}_{\theta} = \left(1 - e^{-\bar{\lambda}}\right)\left(\overline{R}^{f} - R^{m}\right) + \theta\left(1 - e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta} \geq 0$$

and taking a second derivative shows that:

$$\chi^{+}_{\theta\theta} = 2\left(1 - e^{-\bar{\lambda}}\right)\left(\overline{R}^{f}_{\theta}\right) + \theta\left(1 - e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta\theta} \geq 0,$$

which shows that the function is convex as well.

Likewise, using (5), we have that:

$$\chi^{-}_{\theta} = \left(1 - e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta} \geq 0,$$

and furthermore:

$$\chi^{-}_{\theta\theta} = \left(1 - e^{-\bar{\lambda}}\right)\overline{R}^{f}_{\theta\theta} \geq 0.$$

In turn, the spread $\Sigma \equiv \chi^{-} - \chi^{+}$ satisfies:

$$\Sigma \equiv \left(\Psi^{-} - \Psi^{+}\right)\left(\overline{R}^{f} - R^{m}\right) + \left(1 - \Psi^{-}\right)(R^{w} - R^{m})$$

$$= (R^{w} - R^{m})\left[\frac{(\theta + (1-\theta)\exp(\lambda))^{1-\eta} - \theta}{\exp(\lambda)}\right].$$

Thus,

$$\Sigma_{\theta} = -\left[\frac{(1-\eta)(\theta + (1-\theta)\exp(\lambda))^{-\eta}(\exp(\lambda) - 1) + 1}{\exp(\lambda)}\right] < 0,$$

although:

$$\Sigma_{\theta\theta} = -\left[\frac{\eta(1-\eta)(\theta + (1-\theta)\exp(\lambda))^{-(\eta+1)}(\exp(\lambda) - 1)^{2}}{\exp(\lambda)}\right] < 0.$$

Thus, the spread falls and is concave as $\theta < 1$. The result for $\theta > 1$ follows by symmetry.

D. Proof of Derivatives with respect to $\bar{\lambda}$

Recall that:

$$\begin{aligned} \chi^{+} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}{\bar{\theta}-1}\right) \in\left[0, r^{w}-r^{m}\right] \\ \text{and } \chi^{-} &= \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}{\bar{\theta}-1}\right) \in\left[0, r^{w}-r^{m}\right]. \end{aligned}$$

The parameter $\bar{\lambda}$ only enters in $\bar{\theta}$. Thus, we first obtain the derivatives with respect to $\bar{\theta}$. For that, define

$$q^{+}\left(\bar{\theta}\right) \equiv\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}{\bar{\theta}-1}\right) \in[0,1]$$

and

$$q^{-}\left(\bar{\theta}\right) \equiv\left(\frac{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}{\bar{\theta}-1}\right) \in[0,1].$$

We have that:

$$\begin{aligned} q_{\bar{\theta}}^{+}\left(\bar{\theta}\right) &= q^{+}\left(\bar{\theta}\right)\left(\frac{1-\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}}{\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}}-\frac{1}{\bar{\theta}-1}\right) \\ &= q^{+}\left(\bar{\theta}\right)\left(\frac{\bar{\theta}-\eta\bar{\theta}^{\eta}\theta^{1-\eta}-1+\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}-\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)\left(\bar{\theta}-1\right)}\right) \\ &= q^{+}\left(\bar{\theta}\right)\left(\frac{\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}+(1-\eta)\,\bar{\theta}^{\eta}\theta^{1-\eta}-1}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{1-\eta}\right)\left(\bar{\theta}-1\right)}\right). \end{aligned}$$

The denominator is always positive. Hence the sign inherits the sign of the numerator. The numerator satisfies:

$$\eta\bar{\theta}^{\eta-1}\theta^{1-\eta}+(1-\eta)\,\bar{\theta}^{\eta}\theta^{1-\eta}-1 = \bar{\theta}^{\eta}\theta^{1-\eta}\left(1-\eta\left(1-\bar{\theta}^{-1}\right)\right)-1.$$

Thus, the numerator is positive if:

$$\theta^{1-\eta}\left((1-\eta)\,\bar{\theta}+\eta\right)>\bar{\theta}^{1-\eta}.$$

Likewise:

$$\begin{aligned} q_{\bar{\theta}}^{-}\left(\bar{\theta}\right) &= q^{-}\left(\bar{\theta}\right)\left(\frac{1-\eta\bar{\theta}^{\eta-1}\theta^{-\eta}}{\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}}-\frac{1}{\bar{\theta}-1}\right) \\ &= q^{-}\left(\bar{\theta}\right)\left(\frac{\bar{\theta}-\eta\bar{\theta}^{\eta}\theta^{-\eta}-1+\eta\bar{\theta}^{\eta-1}\theta^{-\eta}-\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)\left(\bar{\theta}-1\right)}\right) \\ &= q^{-}\left(\bar{\theta}\right)\left(\frac{\eta\bar{\theta}^{\eta-1}\theta^{-\eta}+(1-\eta)\,\bar{\theta}^{\eta}\theta^{-\eta}-1}{\left(\bar{\theta}-\bar{\theta}^{\eta}\theta^{-\eta}\right)\left(\bar{\theta}-1\right)}\right). \end{aligned}$$

The numerator satisfies:

$$\eta\bar{\theta}^{\eta-1}\theta^{-\eta}+(1-\eta)\,\bar{\theta}^{\eta}\theta^{-\eta}-1 = \bar{\theta}^{\eta}\theta^{-\eta}\left(1-\eta\left(1-\bar{\theta}^{-1}\right)\right)-1.$$

The denominator is always positive. Hence the sign inherits the sign of the numerator. The numerator is positive if:

$$\theta^{-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)>\bar{\theta}^{1-\eta}.$$

We must inspect the validity of the following conditions:

- $\theta^{1-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)-\bar{\theta}^{1-\eta}>0$
- $\theta^{-\eta}\left(\left(1-\eta\right)\bar{\theta}+\eta\right)-\bar{\theta}^{1-\eta}>0$

Again, we break the analysis into cases:

Case $\theta<1$. When $\theta<1$, $\bar{\theta}\in[0,\theta]$. Consider the first condition. Then, at $\bar{\theta}=0$, the condition is satisfied. However, it is violated at $\bar{\theta}=\theta$. By continuity, the derivative changes for some $\bar{\theta}$. This implies that $q_{\bar{\theta}}^{+}\left(\bar{\theta}\right)$ switches sign for some $\bar{\theta}$ when $\theta<1$. Hence, $\chi_{\bar{\lambda}}^{+}$ is not monotone.

Now consider the second condition. Then, at $\bar{\theta}=0$, the condition is satisfied. At $\bar{\theta}=\theta$, the condition is also satisfied since:

$$\eta\left(1-\theta\right)>0.$$

The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{-\eta}-\bar{\theta}^{-\eta}\right)<0,$$

thus, the condition is satisfied for all values. This implies that $q_{\bar{\theta}}^{-}\left(\bar{\theta}\right)>0$ for $\theta<1$. Hence, $\chi_{\bar{\lambda}}^{-}$ is monotone. Since:

$$\chi_{\bar{\lambda}}^{-}=q_{\bar{\theta}}^{-}\left(\bar{\theta}\right)\times\bar{\theta}_{\bar{\lambda}},$$

we have that $\chi_{\bar{\lambda}}^{-}$ is monotone decreasing because $\bar{\theta}_{\bar{\lambda}}<0$.

Case $\theta>1$. When $\theta>1$, $\bar{\theta}\in[\theta,\infty]$. Consider the first condition. Then, at $\bar{\theta}=\theta$, the condition is satisfied. The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{1-\eta}-\bar{\theta}^{-\eta}\right)>0.$$

Hence, the condition is always satisfied. Then, this implies that $q_{\bar{\theta}}^{+}\left(\bar{\theta}\right)>0$ for $\theta>1$. Hence, $\chi_{\bar{\lambda}}^{-}$ is monotone. Since:

$$\chi_{\bar{\lambda}}^{+}=q_{\bar{\theta}}^{+}\left(\bar{\theta}\right)\times\bar{\theta}_{\bar{\lambda}},$$

we have that $\chi_{\bar{\lambda}}^{+}$ is monotone increasing because $\bar{\theta}_{\bar{\lambda}}>0$.

Now consider the second condition. Then, at $\bar{\theta}=\theta$, the condition is violated. The derivative of the term in the left is:

$$\left(1-\eta\right)\left(\theta^{-\eta}-\bar{\theta}^{-\eta}\right)>0,$$

thus, the condition must be satisfied for some values. By continuity, the derivative changes for some $\bar{\theta}$. This implies that $q_{\bar{\theta}}^{-}\left(\bar{\theta}\right)$ switches sign for some $\bar{\theta}$ when $\theta>1$. Hence, $\chi_{\bar{\lambda}}^{-}$ is not monotone.

Case $\theta=1$. In this case, λ does not have an effect on $\bar{R}^{f}=(1-\eta)R^{w}+\eta R^{m}$. However, note that:

$$\chi^{-}=(R^{w}-R^{m})-\Psi^{-}(R^{w}-\overline{R}^{f}),\qquad\chi^{+}=\Psi^{+}(\overline{R}^{f}-R^{m}).\tag{25}$$

Thus,

$$\chi_{\bar{\lambda}}^{-} = -\Psi_{\bar{\lambda}}^{-}(R^{w} - \overline{R}^{f}) = -\eta\Psi_{\bar{\lambda}}^{-}(R^{w} - R^{m}) < 0, \qquad \chi_{\bar{\lambda}}^{+} = \Psi_{\bar{\lambda}}^{+}(\overline{R}^{f} - R^{m}) = (1-\eta)\,\Psi_{\bar{\lambda}}^{+}(R^{w} - R^{m}) > 0.$$

Average Interest Rate. We now consider the derivative of the average interest rate. Recall that the endogenous bargaining power is:

$$\phi(\theta) = \begin{cases} 1 - \frac{\bar{\theta} - \theta^{1-\eta}\bar{\theta}^{\eta}}{\bar{\theta} - \theta} & \text{if } \theta > 1 \\ \eta & \text{if } \theta = 1 \\ 1 - \frac{\theta^{-1} - \theta^{-\eta}\bar{\theta}^{\eta-1}}{\bar{\theta}^{-1} - \theta^{-1}} & \text{if } \theta < 1. \end{cases}$$

For $\theta > 1$, we have that:

$$\phi(\theta) = 1 - q^{+}\left(\bar{\theta}\right).$$

Thus, in this case, we have shown above that:

$$q_{\bar{\theta}}^{+}\left(\bar{\theta}\right) > 0.$$

This implies that

$$\phi_{\lambda}(\theta) = -\frac{\partial}{\partial\bar{\theta}}\left[q^{+}\left(\bar{\theta}\right)\right] \cdot \frac{\partial}{\partial\lambda}\left[\bar{\theta}\right] < 0.$$

For $\theta < 1$, we exploit the symmetry property:

$$\phi(\theta) = 1 - \phi\left(\theta^{-1}\right)$$

Thus,

$$\phi_{\lambda}(\theta) > 0.$$

E. Proof of Proposition 8

Derivative of the dispersion of the interbank market rate w.r.t. θ:

Case $\theta > 1$.

$$\begin{aligned}
Q_{min}^{max} &= max_{\tau}\left\{r_{\tau}^{f}\right\} - min\left\{r_{\tau}^{f}\right\} = r_{1}^{f} - r_{0}^{f} = -((-R^{m} + R^{w})(1-\eta)) \\
&+ \frac{e^{-\lambda}(-R^{m}+R^{w})\eta\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{1-\eta}\right)}{-1+\theta} \\
&+ \frac{e^{-\lambda}(-R^{m}+R^{w})(1-\eta)\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-\eta}\right)}{-1+\theta} \\
&= \frac{-e^{-\lambda}(R^{w}-R^{m})\theta^{-\eta}\left(-\left((-1+\eta)\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\right)+\eta\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta+\left(-1+e^{\lambda}\eta\right)\theta^{\eta}-e^{\lambda}\eta\theta^{1+\eta}\right)}{-1+\theta} \\
&= \frac{e^{-\lambda}(R^{w}-R^{m})\theta^{-\eta}\left(\left((-1+\eta)\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\right)-\eta\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta-\left(-1+e^{\lambda}\eta\right)\theta^{\eta}+e^{\lambda}\eta\theta^{1+\eta}\right)}{\theta-1}
\end{aligned}$$

And:

$$\begin{aligned}
\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial\theta} &= -\frac{e^{-\lambda}(-R^{m}+R^{w})\eta\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{1-\eta}\right)}{(-1+\theta)^{2}} \\
&+ \frac{e^{-\lambda}(-R^{m}+R^{w})(1-\eta)\left(e^{\lambda}+\eta\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-1-\eta}-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}\theta^{-\eta}\right)}{-1+\theta} \\
&- \frac{e^{-\lambda}(-R^{m}+R^{w})(1-\eta)\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-\eta}\right)}{(-1+\theta)^{2}} \\
&+ \frac{e^{-\lambda}(-R^{m}+R^{w})\eta\left(e^{\lambda}-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}\theta^{1-\eta}-(1-\eta)\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-\eta}\right)}{-1+\theta}
\end{aligned}$$

Since $\theta > 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^{w}-R^{m}) > 0$, we have:

$$\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial\theta} = \frac{\underbrace{e^{-\lambda}(R^{m}-R^{w})\theta^{-1-\eta}\left(1+e^{\lambda}(-1+\theta)\right)}_{<0}F(\theta,\lambda,\eta)}{\underbrace{\left(1+e^{\lambda}(-1+\theta)\right)(-1+\theta)^{2}}_{>0}}$$

Where:

$$F(\theta,\lambda,\eta) = \theta^{1+\eta} + \left(1+e^{\lambda}(-1+\theta)\right)^{\eta-1}\left(-\theta+(-1+\theta)\left(e^{\lambda}(-1+\eta)(\eta(-1+\theta)+\theta)+\eta(-1+\eta-\eta\theta)\right)\right)$$

And it's important to mention that this function $F(\theta,\lambda,\eta)$ is increasing in η, so if we take the following limit $\eta \to 1$:

$$F(\theta,\lambda,\eta) < \max F(\theta,\lambda,\eta) \approx \lim_{\eta\to 1} F(\theta,\lambda,\eta) = \theta^{2} + (-\theta + (-1+\theta)(-\theta)) = \theta^{2} - \theta^{2} = 0$$

And therefore:

$$\frac{\partial Q_{\text{min}}^{\text{max}}}{\partial\theta} = \frac{e^{-\lambda}(R^{m}-R^{w})\theta^{-1-\eta}\left(1+e^{\lambda}(-1+\theta)\right)F(\theta,\lambda,\eta)}{\left(1+e^{\lambda}(-1+\theta)\right)(-1+\theta)^{2}} > 0$$

Case $\theta < 1$.

$$Q_{min}^{max} = max_{\tau}\left\{r_{\tau}^{f}\right\} - min\left\{r_{\tau}^{f}\right\} = r_0^f - r_1^f = (-R^m + R^w)(1-\eta) - \frac{(-R^m + R^w)\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$- \frac{(-R^m + R^w)(1-\eta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

Where:

$$\frac{\partial Q_{\min}^{\max}}{\partial\theta} = -\frac{(-R^m + R^w)\eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2}$$

$$- \frac{(-R^m + R^w)(1-\eta)\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}} - \left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2}$$

$$- \frac{(-R^m + R^w)\eta\left(-\frac{-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2} + \eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}\theta^{1-\eta} - (1-\eta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$- \frac{(-R^m + R^w)(1-\eta)\left(-\frac{-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2} + \eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-1-\eta} + \eta\left(-\frac{e^{\lambda}(1-\theta)}{\theta^2} - \frac{e^{\lambda}}{\theta}\right)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}}$$

$$= -\frac{e^{-\lambda}(R^m - R^w)\theta^{-\eta}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta} + \left(-1+e^{\lambda}\right)\eta\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{(-1+\theta)^2}$$

Since $\theta < 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^w - R^m) > 0$, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\theta} = \frac{\underbrace{e^{-\lambda}(R^w - R^m)\theta^{-\eta}\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}}_{>0}\left(-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{\underbrace{(-1+\theta)^2}_{>0}}$$

We need to analyze the second factor of the numerator. So we can begin with:

$$0 < \min(2-\eta+\theta\eta-\theta) = 1 < (2-\eta+\theta\eta-\theta)$$

Also we know that $\left(-1+e^{\lambda}\right)\eta > 0$ and $-1+\theta < 0$. So we have:

$$\left(-1+e^{\lambda}\right)\eta(2-\eta+\theta\eta-\theta)(-1+\theta) < 0$$

$$\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta) + \left(\underbrace{\theta^{\eta}-1}_{<0}\right) < \left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta) < 0$$

And therefore the second factor of the numerator will be:

$$-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta} < 0$$

Finally, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\theta} = \frac{e^{-\lambda}(R^{w}-R^{m})\theta^{-\eta}\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}\left(-1+\left(-1+e^{\lambda}\right)\eta(2+\eta(-1+\theta)-\theta)(-1+\theta)+\theta^{\eta}\right)}{(-1+\theta)^{2}} < 0$$

Case $\theta = 1$. As we can see in the last two cases, if we take the following limit $\theta \to 1$, we have that $\frac{\partial Q_{\min}^{\max}}{\partial\theta} \to 0$. This is because when the market tightness disappears, we have that the interbank rate is always the rate on reserves (a constant).

Derivative of the dispersion of the interbank market rate w.r.t. λ:

Case $\theta > 1$.

$$\begin{aligned}
\frac{\partial Q_{\min}^{\max}}{\partial\lambda} &= -\frac{e^{-\lambda}(-R^{m}+R^{w})\eta\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{1-\eta}\right)}{-1+\theta} \\
&+\frac{e^{-\lambda}(-R^{m}+R^{w})\eta\left(e^{\lambda}(-1+\theta)-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}(-1+\theta)\theta^{1-\eta}\right)}{-1+\theta} \\
&-\frac{e^{-\lambda}(-R^{m}+R^{w})(1-\eta)\left(1+e^{\lambda}(-1+\theta)-\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\theta^{-\eta}\right)}{-1+\theta} \\
&+\frac{e^{-\lambda}(-R^{m}+R^{w})(1-\eta)\left(e^{\lambda}(-1+\theta)-e^{\lambda}\eta\left(1+e^{\lambda}(-1+\theta)\right)^{-1+\eta}(-1+\theta)\theta^{-\eta}\right)}{-1+\theta} \\
&=\frac{e^{-\lambda}(R^{m}-R^{w})\theta^{-\eta}\left(\left(1+e^{\lambda}(-1+\theta)\right)^{\eta}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\left(1+e^{\lambda}(-1+\theta)\right)\theta^{\eta}\right)}{-1+e^{\lambda}(-1+\theta)^{2}+\theta}
\end{aligned}$$

Since $\theta > 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^{w}-R^{m}) > 0$, we have:

$$\frac{\partial Q_{\min}^{\max}}{\partial\lambda} = \frac{\underbrace{e^{-\lambda}(R^{m}-R^{w})\theta^{-\eta}\left(1+e^{\lambda}(-1+\theta)\right)}_{<0}\left(\left(1+e^{\lambda}(-1+\theta)\right)^{\eta-1}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\theta^{\eta}\right)}{\underbrace{-1+e^{\lambda}(-1+\theta)^{2}+\theta}_{>0}}$$

Where:

$$G(\theta,\lambda,\eta) = \left(1+e^{\lambda}(-1+\theta)\right)^{\eta-1}\left(-1+e^{\lambda}(-1+\eta)(-1+\theta)\right)(1+\eta(-1+\theta))+\theta^{\eta}$$

And it's important to mention that the last function is increasing in η, so if we take the following limit $\eta \to 1$:

$$G(\theta,\lambda,\eta) < \max G(\theta,\lambda,\eta) \approx \lim_{\eta\to 1} G(\theta,\lambda,\eta) = \left(1+e^{\lambda}(-1+\theta)\right)^{0}(-1)(\theta)+\theta = -\theta+\theta = 0$$

And therefore:

$$\frac{\partial Q^{\max}_{\min}}{\partial \lambda} = \frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(1+e^{\lambda}(-1+\theta)\right)G(\theta,\lambda,\eta)}{-1+e^{\lambda}(-1+\theta)^2+\theta} > 0$$

Case $\theta < 1$.

$$\begin{aligned}
\frac{\partial Q^{\max}_{\min}}{\partial \lambda} =& -\frac{(-R^m+R^w)(1-\eta)\left(-\frac{e^{\lambda}(1-\theta)}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}+e^{\lambda}\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}(1-\theta)\theta^{-1-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}} \\
&-\frac{e^{\lambda}(-R^m+R^w)\eta(1-\theta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}-\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{1-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta} \\
&-\frac{e^{\lambda}(-R^m+R^w)(1-\eta)(1-\theta)\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}-\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{\eta}\theta^{-\eta}\right)}{\left(-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^2\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta} \\
&-\frac{(-R^m+R^w)\eta\left(-\frac{e^{\lambda}(1-\theta)}{\left(1+\frac{e^{\lambda}(1-\theta)}{\theta}\right)^2\theta}+e^{\lambda}\eta\left(\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}\right)^{1+\eta}(1-\theta)\theta^{-\eta}\right)}{-1+\frac{1}{1+\frac{e^{\lambda}(1-\theta)}{\theta}}} \\
=& \frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(1+\eta(-1+\theta))\left(e^{\lambda}\eta(-1+\theta)-\theta\right)-\theta^{1+\eta}\right)}{-1+\theta}
\end{aligned}$$

Since $\theta < 1$, $\lambda \in [0,1]$, $\eta \in [0,1]$ and $(R^w - R^m) > 0$, we have:

$$\frac{\partial Q^{\max}_{\min}}{\partial \lambda} = \frac{\underbrace{e^{-\lambda}(R^m-R^w)\theta^{-\eta}}_{<0}\left(-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}(1+\eta(-1+\theta))\left(e^{\lambda}\eta(-1+\theta)-\theta\right)-\theta^{1+\eta}\right)}{\underbrace{-1+\theta}_{<0}}$$

Where:

$$H(\theta,\lambda,\eta) = \underbrace{-\left(\frac{1}{1+e^{\lambda}\left(-1+\frac{1}{\theta}\right)}\right)^{\eta}}_{\in(-1,0)}\underbrace{(1+\eta(-1+\theta))}_{\in(0,1)}\underbrace{\left(e^{\lambda}\eta(-1+\theta)-\theta\right)}_{<0}-\underbrace{\theta^{1+\eta}}_{\in(0,1)}$$

All the factors of the first term in the last equation are decreasing w.r.t. η and the second term is also decreasing w.r.t. η. So all the terms are decreasing and bounded, and therefore the limit w.r.t.η is:

$$H(\theta,\lambda,\eta) < \min H(\theta,\lambda,\eta) \approx \lim_{\eta\to 1} H(\theta,\lambda,\eta) = 0$$

And therefore:

$$\frac{\partial Q^{\max}_{\min}}{\partial \lambda} = \frac{e^{-\lambda}(R^m-R^w)\theta^{-\eta}H(\theta,\lambda,\eta)}{-1+\theta} > 0$$

Case $\theta = 1$. As we can see in the last two cases, if we take the following limit $\theta \to 1$, we have that $\frac{\partial Q_{\min}^{\max}}{\partial \lambda} \to 0$.This is because when the market tightness disappears, we have that the interbank rate is always the rate on reserves (a constant).

F. Proof of Proposition 9

Let us study the three possible cases,

As $\theta \to 1$:

Because $\theta = 1$, observe that

$$\Psi^{+} = 1 - e^{-\bar{\lambda}} \qquad \text{and} \qquad \Psi^{-} = 1 - e^{-\bar{\lambda}}.$$

Therefore,

$$\lim_{\theta \to 1} \left\{ \chi^{+} \right\} = \lim_{\theta \to 1} \left\{ (1-\eta)\left(r^{w} - r^{m}\right)\left(1 - e^{-\bar{\lambda}}\right) \right\} = (1-\eta)\left(1 - e^{-\bar{\lambda}}\right)\left(r^{w} - r^{m}\right).$$

Also,

$$\lim_{\theta \to 1} \left\{ \chi^{-} \right\} = \lim_{\theta \to 1} \left\{ \left(r^{w} - r^{m}\right)\left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right) \right\} = \left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right)\left(r^{w} - r^{m}\right).$$

Finally,

$$\lim_{\theta \to 1} \left\{ \overline{r}^{f} \right\} = \lim_{\theta \to 1} \left\{ (1-\eta) r^{w} + \eta r^{m} \right\} = (1-\eta) r^{w} + \eta r^{m}.$$

As $\theta \to \infty$:

Because $\theta > 1$, observe that

$$\Psi^{+} = 1 - e^{-\bar{\lambda}} \qquad \text{and} \qquad \Psi^{-} = 0.$$

Now, before proceeding, realise that

$$\lim_{\theta \to \infty} \left\{ \frac{\bar{\theta}}{\theta} \right\} = \lim_{\theta \to \infty} \left\{ \frac{1 + (\theta - 1)e^{\bar{\lambda}}}{\theta} \right\} = \lim_{\theta \to \infty} \left\{ \frac{1}{\theta} + \left(1 - \frac{1}{\theta}\right) e^{\bar{\lambda}} \right\} = e^{\bar{\lambda}}.$$

Therefore,

$$\begin{aligned}
\lim_{\theta \to \infty} \left\{ \chi^{+} \right\} &= \lim_{\theta \to \infty} \left\{ \left(r^{w} - r^{m}\right) \left(\frac{\bar{\theta}}{\theta}\right)^{\eta} \left(\frac{\theta^{\eta} \bar{\theta}^{1-\eta} - \theta}{\bar{\theta} - 1} \right) \right\} \\
&= \lim_{\theta \to \infty} \left\{ \left(r^{w} - r^{m}\right) \left(\frac{\bar{\theta}}{\theta}\right)^{\eta} \left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta} - 1}{\frac{\bar{\theta}}{\theta} - \frac{1}{\theta}} \right) \right\} \\
&= \left(r^{w} - r^{m}\right) \left(\lim_{\theta \to \infty} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \left(\frac{\left(\lim_{\theta \to \infty} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{1-\eta} - 1}{\lim_{\theta \to \infty} \left\{ \frac{\bar{\theta}}{\theta} \right\} - \lim_{\theta \to \infty} \left\{ \frac{1}{\theta} \right\}} \right) \\
&= \left(r^{w} - r^{m}\right) e^{\eta \bar{\lambda}} \left(\frac{e^{\bar{\lambda}(1-\eta)} - 1}{e^{\bar{\lambda}}} \right) \\
&= \left(r^{w} - r^{m}\right) \left(1 - e^{-\bar{\lambda}(1-\eta)}\right).
\end{aligned}$$

Also,

$$
\begin{aligned}
\lim_{\theta\to\infty}\left\{\chi^-\right\} &= \lim_{\theta\to\infty}\left\{\left(r^w - r^m\right)\left(\frac{\bar\theta}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar\theta^{1-\eta}-1}{\bar\theta-1}\right)\right\} \\
&= \lim_{\theta\to\infty}\left\{\left(r^w - r^m\right)\left(\frac{\bar\theta}{\theta}\right)^\eta\left(\frac{\left(\frac{\bar\theta}{\theta}\right)^{1-\eta}-\frac{1}{\theta}}{\frac{\bar\theta}{\theta}-\frac{1}{\theta}}\right)\right\} \\
&= \left(r^w - r^m\right)\left(\lim_{\theta\to\infty}\left\{\frac{\bar\theta}{\theta}\right\}\right)^\eta\left(\frac{\left(\lim_{\theta\to\infty}\left\{\frac{\bar\theta}{\theta}\right\}\right)^{1-\eta}-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}{\lim_{\theta\to\infty}\left\{\frac{\bar\theta}{\theta}\right\}-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}\right) \\
&= \left(r^w - r^m\right)e^{\eta\bar\lambda}\left(\frac{e^{\bar\lambda(1-\eta)}}{e^{\bar\lambda}}\right) \\
&= r^w - r^m.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\lim_{\theta\to\infty}\left\{\bar r^f\right\} &= \lim_{\theta\to\infty}\left\{r^w - \left(\left(\frac{\bar\theta}{\theta}\right)^\eta - 1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right)\right\} \\
&= r^w - \left(\left(\lim_{\theta\to\infty}\left\{\frac{\bar\theta}{\theta}\right\}\right)^\eta - 1\right)\left(\frac{1}{1-\lim_{\theta\to\infty}\left\{\frac{1}{\theta}\right\}}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) \\
&= r^w - \left(e^{\bar\lambda\eta}-1\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) \\
&= \left(\frac{e^{\bar\lambda}-e^{\bar\lambda\eta}}{e^{\bar\lambda}-1}\right)r^w + \left(\frac{e^{\bar\lambda\eta}-1}{e^{\bar\lambda}-1}\right)r^m.
\end{aligned}
$$

As $\theta \to 0:$

Because $\theta < 1$, observe that

$$
\Psi^+ = 0 \qquad \text{and} \qquad \Psi^- = 1 - e^{-\bar\lambda}.
$$

Now, before proceeding, realise that

$$
\lim_{\theta\to 0}\left\{\frac{\bar\theta}{\theta}\right\} = \lim_{\theta\to 0}\left\{\frac{\frac{\theta}{\theta+(1-\theta)e^{\bar\lambda}}}{\theta}\right\} = \lim_{\theta\to 0}\left\{\frac{1}{\theta+(1-\theta)e^{\bar\lambda}}\right\} = e^{-\bar\lambda}.
$$

Therefore,

$$
\begin{aligned}
\lim_{\theta\to 0}\left\{\chi^+\right\} &= \lim_{\theta\to 0}\left\{\left(r^w - r^m\right)\left(\frac{\bar\theta}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar\theta^{1-\eta}-\theta}{\bar\theta-1}\right)\right\} \\
&= \left(r^w - r^m\right)\left(\lim_{\theta\to 0}\left\{\frac{\bar\theta}{\theta}\right\}\right)^\eta\left(\frac{\lim_{\theta\to 0}\left\{\theta^\eta\bar\theta^{1-\eta}\right\}-\lim_{\theta\to 0}\left\{\theta\right\}}{\lim_{\theta\to 0}\left\{\bar\theta\right\}-1}\right) \\
&= 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\lim_{\theta \to 0} \left\{ \chi^{-} \right\} &= \lim_{\theta \to 0} \left\{ \left(r^{w} - r^{m} \right) \left(\frac{\bar{\theta}}{\theta} \right)^{\eta} \left(\frac{\theta^{\eta} \bar{\theta}^{1-\eta} - 1}{\bar{\theta} - 1} \right) \right\} \\
&= \left(r^{w} - r^{m} \right) \left(\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \left(\frac{\lim_{\theta \to 0} \left\{ \theta^{\eta} \bar{\theta}^{1-\eta} \right\} - 1}{\lim_{\theta \to 0} \left\{ \bar{\theta} \right\} - 1} \right) \\
&= \left(r^{w} - r^{m} \right) e^{-\bar{\lambda} \eta}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\lim_{\theta \to 0} \left\{ \bar{r}^{f} \right\} &= \lim_{\theta \to 0} \left\{ r^{w} - \left(\frac{1}{1-\theta} \right) \left(\frac{\theta}{\bar{\bar{\theta}}} \right) \left(1 - \left(\frac{\bar{\theta}}{\theta} \right)^{\eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \right\} \\
&= r^{w} - \left(\frac{1}{1 - \lim_{\theta \to 0} \left\{ \theta \right\}} \right) \left(\lim_{\theta \to 0} \left\{ \frac{\theta}{\bar{\bar{\theta}}} \right\} \right) \left(1 - \left(\lim_{\theta \to 0} \left\{ \frac{\bar{\theta}}{\theta} \right\} \right)^{\eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= r^{w} - e^{\bar{\lambda}} \left(1 - e^{-\bar{\lambda} \eta} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= r^{w} - \left(e^{\bar{\lambda}} - e^{\bar{\lambda}(1-\eta)} \right) \left(\frac{r^{w} - r^{m}}{e^{\bar{\lambda}} - 1} \right) \\
&= \left(\frac{e^{\bar{\lambda}(1-\eta)} - 1}{e^{\bar{\lambda}} - 1} \right) r^{w} + \left(\frac{e^{\bar{\lambda}} - e^{\bar{\lambda}(1-\eta)}}{e^{\bar{\lambda}} - 1} \right) r^{m}.
\end{aligned}
$$

G. Proof of Proposition 10

G.1 Matching Efficiency

Let us first start with the limiting properties of the matching efficiency parameter.

Let us study the three possible cases,

$\underline{\theta = 1}$:

Because $\theta = 1$, observe that

$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1 - e^{-\bar{\lambda}}\right\} = 1, \qquad \text{and}$$
$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1 - e^{-\bar{\lambda}}\right\} = 1.$$

Therefore,

$$\begin{aligned}\lim_{\bar{\lambda}\to\infty}\left\{\chi^{+}\right\} &= \lim_{\theta\to\infty}\left\{(1-\eta)\left(1 - e^{-\bar{\lambda}}\right)(r^{w} - r^{m})\right\} \\ &= (1-\eta)\left(1 - \lim_{\bar{\lambda}\to\infty}\left\{e^{-\bar{\lambda}}\right\}\right)(r^{w} - r^{m}) \\ &= (r^{w} - r^{m})(1-\eta).\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\bar{\lambda}\to\infty}\left\{\chi^{-}\right\} &= \lim_{\theta\to\infty}\left\{\left(1 - \eta\left(1 - e^{-\bar{\lambda}}\right)\right)(r^{w} - r^{m})\right\} \\ &= \left(1 - \eta\left(1 - \lim_{\bar{\lambda}\to\infty}\left\{e^{-\bar{\lambda}}\right\}\right)\right)(r^{w} - r^{m}) \\ &= (r^{w} - r^{m})(1-\eta).\end{aligned}$$

Finally,

$$\lim_{\bar{\lambda}\to\infty}\left\{\overline{r}^{f}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{(1-\eta)r^{w} + \eta r^{m}\right\} = (1-\eta)r^{w} + \eta r^{m}.$$

$\underline{\theta > 1}$:

Because $\theta > 1$, observe that

$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{1 - e^{-\bar{\lambda}}\right\} = 1, \qquad \text{and}$$
$$\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{1 - e^{-\bar{\lambda}}}{\theta}\right\} = \frac{1}{\theta}.$$

Now, before proceeding, realise that

$$\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{\bar{\bar{\theta}}}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{1 + (\theta - 1)e^{\bar{\lambda}}}\right\} = 0.$$

Therefore,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{+}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{1-\left(\frac{\theta}{\bar{\theta}}\right)^{1-\eta}}{1-\frac{1}{\bar{\theta}}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\frac{1-\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta}{\bar{\theta}}\right\}\right)^{1-\eta}}{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\bar{\theta}}\right\}}\right) \\
&= r^{w}-r^{m}.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^{-}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^{w}-r^{m})\left(\frac{1-\frac{1}{\theta^{\eta}\bar{\theta}^{1-\eta}}}{1-\frac{1}{\bar{\theta}}}\right)\right\} \\
&= (r^{w}-r^{m})\left(\frac{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\theta^{\eta}\bar{\theta}^{1-\eta}}\right\}}{1-\lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\bar{\theta}}\right\}}\right) \\
&= r^{w}-r^{m}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\bar{r}^{f}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-\left(\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}-1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^{w}-r^{m}}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\frac{\left(\frac{1+(\theta-1)e^{\bar{\lambda}}}{\theta}\right)^{\eta}-1}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\left(\frac{1+(\theta-1)e^{\bar{\lambda}}}{\theta}\right)^{\eta}-1}{e^{\bar{\lambda}}-1}\right\}\right) \\
&= r^{w}-(r^{w}-r^{m})\left(\frac{\theta}{\theta-1}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\eta\left(\frac{\theta-1}{\theta}\right)\left(\frac{\theta}{1+(\theta-1)e^{\bar{\lambda}}}\right)^{1-\eta}\right\}\right) \\
&= r^{w}.
\end{aligned}$$

$\underline{\theta < 1}$:
Because $\theta > 1$, observe that

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{+}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{\theta\left(1-e^{-\bar{\lambda}}\right)\right\} = \theta, \qquad \text{and} \\
\lim_{\bar{\lambda}\to\infty}\left\{\Psi^{-}\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{1-e^{-\bar{\lambda}}\right\} = 1.
\end{aligned}$$

Now, before proceeding, realise that

$$\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{\frac{1}{1+\left(\frac{1-\theta}{\theta}\right)e^{\bar{\lambda}}}}{\theta}\right\} = \lim_{\bar{\lambda}\to\infty}\left\{\frac{1}{\theta+(1-\theta)e^{\bar{\lambda}}}\right\} = 0.$$

Therefore,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^+\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^w - r^m)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^w - r^m)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-1}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\} \\
&= (r^w - r^m)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-1}{\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\frac{1}{\theta}}\right) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\chi^-\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{(r^w - r^m)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{(r^w - r^m)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\left(\frac{\bar{\theta}}{\theta}\right)^{1-\eta}-\frac{1}{\theta}}{\frac{\bar{\theta}}{\theta}-\frac{1}{\theta}}\right)\right\} \\
&= (r^w - r^m)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\left(\frac{\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{1-\eta}-\frac{1}{\theta}}{\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}-\frac{1}{\theta}}\right) \\
&= 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\left\{\overline{r}^f\right\} &= \lim_{\bar{\lambda}\to\infty}\left\{r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar{\theta}}\right)\left(1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{r^w - r^m}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= \lim_{\bar{\lambda}\to\infty}\left\{r^w - \left(\frac{r^w - r^m}{1-\theta}\right)\left(1-\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\right)\left(\frac{\theta+(1-\theta)e^{\bar{\lambda}}}{e^{\bar{\lambda}}-1}\right)\right\} \\
&= r^w - \left(\frac{r^w - r^m}{1-\theta}\right)\left(1-\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\bar{\theta}}{\theta}\right\}\right)^{\eta}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{\frac{\theta+(1-\theta)e^{\bar{\lambda}}}{e^{\bar{\lambda}}-1}\right\}\right) \\
&= r^w - \left(\frac{r^w - r^m}{1-\theta}\right)\left(\lim_{\bar{\lambda}\to\infty}\left\{1-\theta\right\}\right) \\
&= r^m.
\end{aligned}$$

G.2 Bargaining Power

Now, let us continue with the limiting properties of the bargaining power parameter.

Let us study the two possible cases,

As $\eta \to 1$ **:**

Firstly, realise that

$$\begin{aligned}\lim_{\eta\to 1}\left\{\chi^{+}\right\} &= \begin{cases}\lim_{\eta\to 1}\left\{(1-\eta)\left(r^{w}-r^{m}\right)\left(1-e^{-\bar{\lambda}}\right)\right\} & \text{if } \theta=1\\ \lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-\theta}{\bar{\theta}-1}\right)\right\} & \text{if } \theta\neq 1\end{cases}\\ &= 0.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\eta\to 1}\left\{\chi^{-}\right\} &= \begin{cases}\lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(1-\eta\left(1-e^{-\bar{\lambda}}\right)\right)\right\} & \text{if } \theta=1\\ \lim_{\eta\to 1}\left\{\left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}^{1-\eta}-1}{\bar{\theta}-1}\right)\right\} & \text{if } \theta\neq 1\end{cases}\\ &= \begin{cases}\left(r^{w}-r^{m}\right)e^{\bar{\lambda}} & \text{if } \theta=1\\ \left(r^{w}-r^{m}\right)\left(\frac{\bar{\theta}}{\theta}\right)\left(\frac{\theta-1}{\bar{\theta}-1}\right) & \text{if } \theta\neq 1\end{cases}\\ &= \begin{cases}\left(r^{w}-r^{m}\right)\left(1-\left(1-e^{\bar{\lambda}}\right)\right) & \text{if } \theta=1\\ \left(r^{w}-r^{m}\right)\left(1-\left(\frac{1-e^{-\bar{\lambda}}}{\theta}\right)\right) & \text{if } \theta>1\\ \left(r^{w}-r^{m}\right)\left(1-\left(1-e^{-\bar{\lambda}}\right)\right) & \text{if } \theta<1\end{cases}\\ &= \left(1-\Psi^{-}\right)\left(r^{w}-r^{m}\right).\end{aligned}$$

Finally,

$$\lim_{\eta\to1}\left\{\overline{r}^f\right\} = \begin{cases} \lim_{\eta\to1}\left\{(1-\eta)r^w - \eta r^m\right\} & \text{if } \theta=1 \\ \lim_{\eta\to1}\left\{r^w - \left(\left(\frac{\bar\theta}{\theta}\right)^\eta - 1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right)\right\} & \text{if } \theta>1 \\ \lim_{\eta\to1}\left\{r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar\theta}\right)\left(1-\left(\frac{\bar\theta}{\theta}\right)^\eta\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right)\right\} & \text{if } \theta<1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta=1 \\ r^w - \left(\frac{\bar\theta-\theta}{\theta}\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta>1 \\ r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar\theta}\right)\left(\frac{\theta-\bar\theta}{\theta}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta<1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta=1 \\ r^w - \left(\frac{\bar\theta-\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta>1 \\ r^w - \left(\frac{1}{\bar\theta}\right)\left(\frac{\theta-\bar\theta}{1-\theta}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta<1 \end{cases}$$

$$= \begin{cases} r^m & \text{if } \theta=1 \\ r^w - \left(\frac{1+(\theta-1)e^{\bar\lambda}-\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta>1 \\ r^w - \left(\frac{\theta}{\theta}\right)\left(\frac{(\theta+(1-\theta)e^{\bar\lambda}-1)}{(1-\theta)\left(\theta+(1-\theta)e^{\bar\lambda}\right)}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right) & \text{if } \theta<1 \end{cases}$$

$$= r^m.$$

As $\eta \to 0$:

Firstly, realize that

$$\lim_{\eta\to0}\left\{\chi^+\right\} = \begin{cases} \lim_{\eta\to0}\left\{(1-\eta)\left(r^w-r^m\right)\left(1-e^{-\bar\lambda}\right)\right\} & \text{if } \theta=1 \\ \lim_{\eta\to0}\left\{\left(r^w-r^m\right)\left(\frac{\bar\theta}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar\theta^{1-\eta}-\theta}{\bar\theta-1}\right)\right\} & \text{if } \theta\neq1 \end{cases}$$

$$= \Psi^+\left(r^w-r^m\right).$$

Also,

$$\lim_{\eta\to0}\left\{\chi^-\right\} = \begin{cases} \lim_{\eta\to0}\left\{\left(r^w-r^m\right)\left(1-\eta\left(1-e^{-\bar\lambda}\right)\right)\right\} & \text{if } \theta=1 \\ \lim_{\eta\to0}\left\{\left(r^w-r^m\right)\left(\frac{\bar\theta}{\theta}\right)^\eta\left(\frac{\theta^\eta\bar\theta^{1-\eta}-1}{\bar\theta-1}\right)\right\} & \text{if } \theta\neq1 \end{cases}$$

$$= r^w - r^m.$$

Finally,

$$\lim_{\eta\to0}\left\{\overline{r}^f\right\} = \begin{cases} \lim_{\eta\to0}\left\{(1-\eta)r^w - \eta r^m\right\} & \text{if } \theta=1 \\ \lim_{\eta\to0}\left\{r^w - \left(\left(\frac{\bar\theta}{\theta}\right)^\eta - 1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right)\right\} & \text{if } \theta>1 \\ \lim_{\eta\to0}\left\{r^w - \left(\frac{1}{1-\theta}\right)\left(\frac{\theta}{\bar\theta}\right)\left(1-\left(\frac{\bar\theta}{\theta}\right)^\eta\right)\left(\frac{r^w-r^m}{e^{\bar\lambda}-1}\right)\right\} & \text{if } \theta<1 \end{cases}$$

$$= r^w.$$

H. Solution of the Portfolio Problem

H.1 Pricing Conditions

H.1.1 Risk Aversion Case

We have the following problem:

$$\max_A \left(\mathbb{E}_{X,\omega} \left[\left(R^m e + \textstyle\sum_{i\in\mathbb{I}} \underbrace{\left(R^i\left(X_t\right) - R^m\right)}_{\text{Liquidity Premium}} \bar{a}^i_t + \underbrace{\chi_{t+1} \left(s \left(\{\bar{a}\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}^i_t \right) \right)}_{\text{Liquidity Yield}} \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}$$

subject to $\Gamma_t \cdot A_t \geq 0$. So the first order condition is:

$$a^i : \mathbb{E}_{X,\omega} \left[(1-\gamma)\, \mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} \right] \right) \right] = 0$$

$$\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} \right] \right) \right] = 0$$

Taking the second term of the expression that is in parentheses, to the right hand side:

$$\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] \right] = -\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega} \left[\chi_s \tfrac{\partial s}{\partial a^i} \right] \right]$$

If we take into account the covariance formula, we have:

$$\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, R^i(X) - R^m \right] = -\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} \right] - \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} \right]$$

$$\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] = -\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} \right] - \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} \right] - \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, R^i(X) - R^m \right]$$

Now, we can obtain the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\frac{-\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

H.1.2 Risk Neutral Case

We have the same problem as the previous case, but considering $\gamma=0$:

$$\max_A\left(\mathbb{E}_{X,\omega}\left[R^m e+\sum\nolimits_{i\in\mathbb{I}}\underbrace{\left(R^i\left(X_t\right)-R^m\right)}_{\text{Liquidity Premium}}\bar{a}^i_t+\underbrace{\chi_{t+1}\left(s\left(\{\bar{a}\}_{i\in\mathbb{I}},e-\sum_{i\in\mathbb{I}}\bar{a}^i_t\right)\right)}_{\text{Liquidity Yield}}\right]\right)$$

subject to $\Gamma_t\cdot A_t\geq 0$. So the first order condition is:

$$a^i:\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]+\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]=0$$
$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]\right]+\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]=0$$

So, now we have the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]\right]$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]$$
$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]$$

H.2 Efficiency

H.2.1 Risk Aversion Case

We have the same problem as the pricing conditions case, so we will follow the same steps to solve it:

$$\max_A \left(\mathbb{E}_{X,\omega} \left[\left(R^m e + \sum_{i\in\mathbb{I}} \underbrace{\left(R^i\left(X_t\right) - R^m\right)}_{\text{Liquidity Premium}} \bar{a}^i_t + \underbrace{\chi_{t+1} \left(s\left(\left\{\bar{a}\right\}_{i\in\mathbb{I}}, e - \sum_{i\in\mathbb{I}} \bar{a}^i_t \right), \theta(\left\{\bar{a}\right\}_{i\in\mathbb{I}}) \right)}_{\text{Liquidity Yield}} \right)^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}$$

subject to $\Gamma_t \cdot A_t \geq 0$. The first order condition is:

$$a^i : \mathbb{E}_{X,\omega} \left[(1-\gamma)\, \mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \right) \right] = 0$$

$$\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \left(\mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] \right) \right] = 0$$

Reordering the last expression:

$$\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] \right] = -\mathbb{E}_{X,\omega} \left[\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \left(\chi_s \tfrac{\partial s}{\partial a^i} + \chi_\theta \tfrac{\partial \theta}{\partial a^i} \right) \right] \right]$$

And using the covariance formula:

$$\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] + \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, R^i(X) - R^m \right] = -\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right]$$
$$- \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right]$$

$$\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_X \left[R^i\left(X_t\right) - R^m \right] = -\mathbb{E}_{X,\omega} \left[(R^e)^{-\gamma} \right] \mathbb{E}_{X,\omega} \left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right] - \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, \chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i} \right]$$
$$- \mathbb{COV}_{X,\omega} \left[(R^e)^{-\gamma}, R^i(X) - R^m \right]$$

Finally, we can obtain the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right)-R^m\right]=-\frac{-\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)-R^m\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}+\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]-\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$
$$-\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right]-R^m=-\left(\mathbb{E}_{X,\omega}\left[\chi_s\frac{\partial s}{\partial a^i}\right]+\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},R^i\left(X\right)+\chi_s\frac{\partial s}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}\right)$$
$$-\left(\mathbb{E}_{X,\omega}\left[\chi_\theta\frac{\partial \theta}{\partial a^i}\right]+\frac{\mathbb{COV}_{X,\omega}\left[\left(R^e\right)^{-\gamma},\chi_\theta\frac{\partial \theta}{\partial a^i}\right]}{\mathbb{E}_{X,\omega}\left[\left(R^e\right)^{-\gamma}\right]}\right)$$

H.2.2 Risk Neutral Case

Now we are going to solve the las problem considering $\gamma=0$:

$$\max_A\left(\mathbb{E}_{X,\omega}\left[R^m e+\sum_{i\in\mathbb{I}}\underbrace{\left(R^i\left(X_t\right)-R^m\right)}_{\text{Liquidity Premium}}\bar{a}^i_t+\underbrace{\chi_{t+1}\left(s\left(\{\bar{a}\}_{i\in\mathbb{I}},e-\sum_{i\in\mathbb{I}}\bar{a}^i_t\right),\theta(\{\bar{a}\}_{i\in\mathbb{I}})\right)}_{\text{Liquidity Yield}}\right]\right)$$

subject to $\Gamma_t \cdot A_t \geq 0$. The first order condition is:

$$a^i : \mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] + \mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right] = 0$$

$$\mathbb{E}_{X,\omega}\left[\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right]\right] + \mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right] = 0$$

And finally we have the asset premium:

$$\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] = -\mathbb{E}_{X,\omega}\left[\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]\right]$$

$$\mathbb{E}_X\left[R^i\left(X_t\right) - R^m\right] = -\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i} + \chi_\theta \frac{\partial \theta}{\partial a^i}\right]$$

$$\mathbb{E}_X\left[R^i\left(X_t\right)\right] - R^m = -\mathbb{E}_{X,\omega}\left[\chi_s \frac{\partial s}{\partial a^i}\right] - \mathbb{E}_{X,\omega}\left[\chi_\theta \frac{\partial \theta}{\partial a^i}\right]$$