Lecture 4 - Asymmetric Information in Macroeconomic Models

Limited Diversification - Risk Insurance

1 Overview

These notes present a simplified version of the model of Eisfeldt (2004, JF). By altering the timing assumptions, I simplify the orginial model Eisfeldt (2004, JF) to attain aggregation while retaining the main insights from that model. The main feature of this model is that the presence of asymmetric information about the returns to investment projects may make investment projects illiquid. This form illiquidity will induce limited diversification by those investors—investors will bear unneccesary idiosyncratic risk. This is something that prevents full risk-insurance. By introducing aggregate risk, this models shows how the lack of diversification may actually affect the aggregate amount of investment —once preferences differ from a log-utility specification.

2 Environment

Timing. Time is discrete and the horizon infinite. Every period is subdivided into two sub-periods. The first sub-period is called the investment stage and the second is called the trading stage. Let t denote a particular period and $s \in \{i, t\}$ the corresponding stage. If needed, I use $x_{t,s}$ to refer to a variable x at time t in stage s, otherwise I simply use the time subscript.

Demographics. There is continuum of agents of measure one.[1] The index $z \in [0, 1]$ denotes a particular individual's identity.

Preferences. Agents agents order consuption streams according to:

$$\mathbb{E}\left[\sum_{t=\tau}^{\infty} \beta^t \frac{c_t^{1-\sigma}(z)}{1-\sigma}\right]$$

where β is the discount factor. Consumption occurs only during the investment stages.

Investing Technology. The technology has several features. There is a risky technology that requires an initial investment cost of φ_t per unit of time and that generates a project with random payoff. This investment is carried out during the investment stage of every period. The random payoff is realized only during the investment stage of the following period —recall there are two sub-periods after. The payoff of these projects is $y_t(z) \in \{0, 1\}$ units of consumption per invested unit —all projects of managed by the same

[1] In Andrea Esfeldt's original version, there is a stochastic death parameter δ used to obtain stationarity. In addition there is a continuum of "buffer agents" that can purchase claims but don't have access to the investment technology to stabilize equilibria. These elements are inessential as we will see below.

agent will yield the same return. I denote by $\tilde{k}_{t+1}(z)$ the number of investment projects invested by agent z at time t.

During the trading stage, the agent receives a signal about the future return to these projects. The signal is a probability that the return is one. The signals are $\pi_t(z) \in \{\pi_1, \pi_2, ..., \pi_n\}$. By assumption, $\pi_m > \pi_{m-1}$ for $m \in \{1, 2, ..., n\}$. In addition, I assume $\pi_n < 1$. Thus, π_m is the probability that the projects return equals 1.

In addition, there is a discrete probability distribution Π_t. The vector Π_t determines the ex-ante probability distribution of obtaining a signal π_m at time t. Thus, $\Pi_t(\pi^m)$ is the fraction of projects with signal π^m. Let's define the ex-ante expected returns, from the perspective of the investment stage given a signal π_n as:

$$R_t^m \equiv \pi^m \frac{1}{\varphi_t}.$$

The ex-ante expected return at the investment stage is:

$$R_t^I = \sum_{m=1}^{n} \Pi(\pi^m) R_t^m.$$

Information. The signal $\pi_t(z)$ is private information to z, the agent that receives the signal.

Aggregate Shocks. There are two aggregate shocks: the cost of investment φ_t and the entire distribution Π_t —the distribution over signals is time varying. Let X_t be the vector of these exogenous state variables.

Asset Markets. There is a single asset market for claims against the return of project. Thus, although in principle agents could ex-ante trade claims against their realization of output, prior to the realization of the signal, this option is ruled.[2] Thus, the asset markets opens only during the trading stage of every period. At this stage, agents can choose to sell a fraction of its risky projects and buy an portfolio index of the projects sold by other investors.

I assume that the portfolio index is constructed by an *obscure* —but benevolent— financial entity left outside of the model. Exploiting the law of large numbers, the entity can sell a riskless claim to a future unit of consumption.

The price for sold projects is p_t^s. The price of the riskless claim is p_t^b. Note that the presence of an Innada condition on the agent's utility implies that it will always demand a minimum amount of the index —no matter what its signals is unless $\pi_t(z) = 1$. Thus, all agents will want to sell at least a small portion of the

[2]Other frictions could lead to this market shutdown. Otherwise, one can assume that the signals are realized ex-ante but this would complicate the analysis because investment rates would differ by signal.

investments —in particular, agents with high signals will always participate.

I denote by $s_t(z)$ the number of projects sold by the agent during the trading stage of period t. Thus, $k_{t+1}(z) = \tilde{k}_{t+1}(z) - s_t(z)$. The amount of purchased riskless assets is $b_{t+1}(z)$. The budget constraint for the agent during the trading stage is:

$$p_t^b b_{t+1}(z) = p_t^s s_t(z).$$

Thus, he sells shares to his projects to buy risk-free asset. The amount of resources held by the agent the following day is:

$$w_{t+1}(z) = b_{t+1}(z) + y_{t+1}(z) k_{t+1}(z).$$

Differences with the Original Paper. There are several differences with Eisfeldt's original specification: First, in the original paper, the timing is different because there are no subperiods. Projects mature two periods ahead and agents carry two vintages of projects. Second, there is a constant endowment e and storage technology. Third, there are stochastic deaths. Fourth, there is access to storage. Finally, there are no outside investors. I have dropped all of these assumptions.

In Eisfeldt's original model, agent's choose to resell their assets to for risk insurance and to exploit investment opportunities. An aggravation of asymmetric information will reduce the amount of risk-insurance, as we show in these notes. There is a second channel that operates when $\sigma \neq 1$. The lack of diversification will reduce the certainty equivalent returns to investment. This is like lowering the returns to investment in a world without risk, which implies that the amount of investment will increase or fall with asymmetric information depending on whether the income or substitution effects dominate the other. In Eisfeldt's original version, the agent's signals and vintage of projects will also affect investment, for the same reason. However, in that model, the presence of endowments implies that this effect is coupled with an effective risk-aversion that varies with the level of wealth. The distribution of wealth is an important state variable in that model. It isn't in this model.

2.1 Recursive Formulation

Let me write the agent's problem in recursive form. I will skip several details because they are similar to the ones that are a recurring theme in these lectures. The value function corresponding to the investment stage is:

$$V_t^i\left(b,y,k,X\right)=\max_{\left\{c,\tilde{k}\right\}\in\mathbb{R}_+^2}\frac{c^{1-\sigma}}{1-\sigma}+E\left[V_t^t\left(\tilde{k},\pi,\tilde{X}\right)\right]$$

$$c+\varphi\tilde{k}=b+yk.$$

The value function during the trading stage is:

$$V_t^t\left(\tilde{k},\pi,\tilde{X}\right)=\max_{s\in\left[0,\tilde{k}\right]}\beta E\left[V_{t+1}^i\left(b',y',k',X\right)|\pi\right] \tag{1}$$

subject to:

$$k'=\tilde{k}-s \text{ and } p_t^b b'=p_t^s s.$$

Notice where is it that π enters in the value functions. Combining the two constraints leads to a familiar single budget constraint:

$$k'+\frac{p_t^b}{p_t^s}b'=\tilde{k}.$$

2.2 Adverse-Selection Price

We assume that finacial markets are anonymous and competitive. The financial institution pooling assets together will purchase all the claims from the agents, and once it pools all these assets together. In turn, it will resell all the pooled resources —under full commitment. Thus, the resource constraint is:

$$\underbrace{\int_0^1 b_{t+1}\left(z\right)dz}_{\substack{\text{Demand for}\\ \text{Risk-less claims}}}=\underbrace{\int_0^1 \pi_t\left(z\right)s_t\left(z\right)dz}_{\text{Supply of Funds}}$$

where $\pi_t\left(z\right)$ is consistent with z's signal and Π. This condition implies that the total resources pooled by the financial institution are the source of funds to honor the returns to the sold indices. Now, a zero profit for the financial intermediary implies that:

$$p_t^b\int_0^1 b_{t+1}\left(z\right)dz=p_t^s\int_0^1 s_t\left(z\right)dz.$$

This condition will translate into a relationship between sold qualities given the different signals and the equilibrium prices. We will characterize p_t^s and p_t^b below using these conditions.

3 Equilibria

An economy is defined by the set of parameters $E = \{\sigma, \beta, \pi_1, \pi_2, ..., \pi_n\}$. The characterization that follows shows that this environment is non-stationary in all of its variables. In particular, the distribution of wealth is wandering off. Nevertheless, we can still define a stationary equilibrium in terms of macroecoomic variables because the economy features stable prices and policy functions. I use the following definition of equilibrium hereafter:

Definition 1. *A stationary recursive competitive equilibrium for an economy E consists of policy functions $\left\{c(w,X), \tilde{k}(w,X), s(w,\pi,X), b(w,\pi,X)\right\}$, a price vector $\bar{p} \equiv \{p^b, p^s\}$, and a sequence of wealth distributions $\lambda_t(w)$, such that:*

1. *The policy functions $\left\{c(w,X), \tilde{k}(w,X), s(w,\pi,X), b(w,\pi,X)\right\}$ solve the problem of an agent with wealth w, given $\bar{p}$.*

2. *$\{p^b, p^s\}$ are consistent with zero-profits by the financial intermediary.*

3. *There is market clearing in asset markets at all t:*

$$\sum_{m=1}^{n} \Pi(\pi^n) \left[\int_0^1 b(w,\pi^n,X_t)\,\lambda_t(w)\,dw\right] = \sum_{m=1}^{n} \Pi(\pi^n) \left[\int_0^1 \pi^n s(w,\pi^n,X_t)\,\lambda_t(w)\,dw\right].$$

4. *The goods market clears at all t:*

$$\int \left(c(w,X_t) + \varphi_t \tilde{k}(w,X_t)\right) \lambda_t(w)\,dw = \varphi_t R_t^I \int \tilde{k}(w)\,\lambda_{t-1}(w)\,dw.$$

5. *The sequence of wealth distributions $\lambda_t(w)$, are consistent with a transition matrix $\Lambda(w'|w)$ induced by the policy functions $\left\{c(w,X), \tilde{k}(w,X), s(w,X), b(w,X)\right\}$, the process for payoffs and signals and the sequence of X_t.*

4 Characterization

Let's characterize equilibria for $\sigma = 1$, that is, under log utility. This case is important because it implies that the scale of investment will not be distorted in response to asymmetric information, or changes in Π_t, and will only depend on φ_t as in a model with perfect information. This is a very useful benchmark. Thus, asymmetric information will only impinge in the distribution of wealth and risk-insurance but will not affect the growth rate of the economy at all. We need to do some work to characterize this economy. Let's proceed by small steps. I enumerate these steps:

1. First, we need to show that for a pair of constant prices $\left\{p^b(X), p^s(X)\right\}$,

$$V_t^i(b,y,k) = V^i(b,y,k) = V^i(w),$$

that is, the actual composition of the agents wealth does not matter. Showing this implies that our definition is internally consistent.

2. Second, we want to show that

$$V^i(w,X) = v^i(X) + \log w$$

and

$$\begin{aligned} c(w) &= (1-\beta)w \\ \varphi\tilde{k}(w) &= \beta w \end{aligned}$$

and moreover,

$$\begin{aligned} s(w) &= \left(1-\omega^k(\pi,X)\right)\beta w \\ k'(w) &= \omega^k(\pi,X)\beta w \end{aligned}$$

where $\omega^k(\pi,X)$ solves the following portfolio problem:

$$\begin{aligned} \Omega(\pi,d(X)) &= \max_{\omega^k\in[0,1]} \exp\left[E\left[\log\left((1-d)\left(1-\omega^k\right)+y'\omega^k\right)\right]|\pi\right] \\ &= \max_{\omega^k\in[0,1]} \exp\left[\pi\log\left((1-d)\left(1-\omega^k\right)+\omega^k\right)+(1-\pi)\log\left((1-d)\left(1-\omega^k\right)\right)\right] \\ &= \max_{\omega^k\in[0,1]} \exp\left[\pi\log\left(1-d\left(1-\omega^k\right)\right)+(1-\pi)\log\left((1-d)\left(1-\omega^k\right)\right)\right] \end{aligned}$$

where $(1-d(X)) = \frac{p^s(X)}{p^b(X)}$ for pairs of prices $p^s(X)$ and $p^b(X)$. This problem is solved for some arbitrary d. Now, $k'(w) = \omega^k(\pi)\beta w$, implies $s(w) = \left(1-\omega^k(\pi)\right)\beta w$. Then, $b(\omega) = \frac{p^s(X)}{p^b(X)}\left(1-\omega^k(\pi)\right)\beta w$. Note that the binary structure of the model yields a closed form solution for $\omega^k(\pi)$. Taking the FOC yields:

$$\begin{aligned} \pi\left[\frac{d}{1-d\left(1-\omega^k\right)}\right] &= (1-\pi)\frac{(1-d)}{(1-d)\left(1-\omega^k\right)} \rightarrow \\ d\left(1-\omega^k\right) &= \frac{(1-\pi)}{\pi}\left(1-d\left(1-\omega^k\right)\right) \rightarrow \\ \omega^k &= \left(1-\frac{(1-\pi)}{d}\right) \end{aligned}$$

but since there are short-selling constraints:

$$\omega^k = \max\left(1 - \frac{(1-\pi)}{d}, 0\right).$$

Notice that when $\pi = 1$, $\omega^k = 1$.

3. Then, we want to argue that since the solutions are indepent of w, that we can obtain prices for a single wealth level w and a given X. Thus, market clearing can be replaced by:

$$\frac{p^s(X)}{p^b(X)} \underbrace{\sum_{m=1}^{n} \Pi_t(\pi^m)\left(1-\omega^k(\pi^n, X)\right)}_{\text{Demand of Risk-Free Investments}} = \underbrace{\sum_{m=1}^{n} \Pi_t(\pi^n)\,\pi^n\left(1-\omega^k(\pi^n)\right)}_{\text{Supply of Goods}}$$

which implies

$$(1-d(X)) = \frac{\sum_{m=1}^{n} \Pi_t(\pi^m)\,\pi^m\left(1-\omega^k(\pi^m, X)\right)}{\sum_{m=1}^{n} \Pi_t(\pi^m)\left(1-\omega^k(\pi^m, X)\right)},$$

or

$$d(X) = \frac{\sum_{m=1}^{n} (1-\pi^m)\,\Pi_t(\pi^m)\left(1-\omega^k(\pi^m, X)\right)}{\sum_{m=1}^{n} \Pi_t(\pi^m)\left(1-\omega^k(\pi^m)\right)}.$$

Now, recall that $\left(1-\omega^k(\pi^m)\right) = \min\left(\frac{(1-\pi^m)}{d}, 1\right)$. Then, $d(X)$ solves:

$$d(X) = \frac{\sum_{m=1}^{n} (1-\pi^m)\,\Pi_t(\pi^m)\min\left((1-\pi^m), d(X)\right)}{\sum_{m=1}^{n} \Pi_t(\pi^m)\min\left((1-\pi^m), d(X)\right)}$$

First Best. There are several notions of first best. Noticeably, an ex-ante first best in which agents trade $\pi-$contingent securities. In that case, each the agent with wealth $w's$ consumption should equal:

$$\bar{c}^1(w) = \left(\sum_{m=1}^{n} R_t^I \varphi_t\right) w.$$

This is the amount of consumption that pools all agents resources together.

Second Best. There is a second best consumption. The one in which trade occurs once signals are known. In that case, consumption depends on the signal and equals:

$$\bar{c}^2(w, \pi_n) = \left(\sum_{m=1}^{n} R_t^n \varphi_t\right) w$$

Notice that ex-ante, the utility of this agent is lower than the utility of the agent under the first best. The equilibrium with asymmetric information features more volatile consumption than either than under the second and first best.

Liquidity. Recall that there are gains from trade in all claims to s. Thus, a measure of the illiquidity in the model is:

$$\sum_{m=1}^{n} \Pi_t (\pi^m) \omega^k (\pi^m, X).$$

Welfare. Clearly, welfare is decreasing in shocks that raise $\omega^k (\pi^n, X)$, that it, welfare decreases with illiquidity in the equilibrium.

Cycles in Wealth Distribution. Shocks that raise ω^k will also lead to a wider distribution of wealth. There is an interesting trade off to investigate. Agents at the lower end of the quality of projects will have incentives to sell all their assets. Their investments are too risky and can sell them at a higher than the fair amount. Agents with low π will grow at the same rate. Agents with very high projects, will not sell their assets, but their incentives to sell assets should be lower because they face low idiosyncratic risk. Thus, they are likely to grow regardless —do to the high returns to their investments. Agents that experience high volatility in returns are those at the middle. The reason is that they will not want to insure against by buying the risk-less assets because of the lemons problem. They will not receive an actuarilly fair price for their assets, and therefore will insure less far from the optimum, yet, π is not large enough to guarantee a low amount of risk.

Shocks that aggravate adverse selection will induce cycles in the dispersion of wealth growth.

4.1 Algorithm

The following algorithm can be used to compute equilibria under log preferences.

1. For each $\pi \in \{\pi_1, \pi_2, ..., \pi_n\}$, solve $\Omega(\pi, d)$ for some arbitary $d = \delta$. Call the sulutions $\tilde{\omega}^k (\pi, \delta)$.
2. Find the fixed point $\frac{\sum_{m=1}^{n} (1-\pi^n) \Pi_t(\pi^n) \left(1-\tilde{\omega}^k(\pi^n,\delta)\right)}{\sum_{m=1}^{n} \Pi_t(\pi^n) \left(1-\tilde{\omega}^k(\pi^n,\delta)\right)} = \delta.$

An equilibrium is $d = \delta$, $\omega^k(\pi) = \omega^k(\pi, \delta)$ and the values for $\tilde{k}$ and (k, s) that correspond to these solutions. We can perform comparative statics with these solutions. Notice importantly that the solutions are independent of the value of φ_t.

4.2 Properties of Equilibria under $\sigma = 1$

Example 1. Let me now present some simple examples. First, consider an example where $\Pi_t = [0.2; 0.3; 0.5]$ so that the distribution is constant over time. Also, assume that π_1, π_2 and π_3 are 0.7, 0.9 and 0.95. Then,

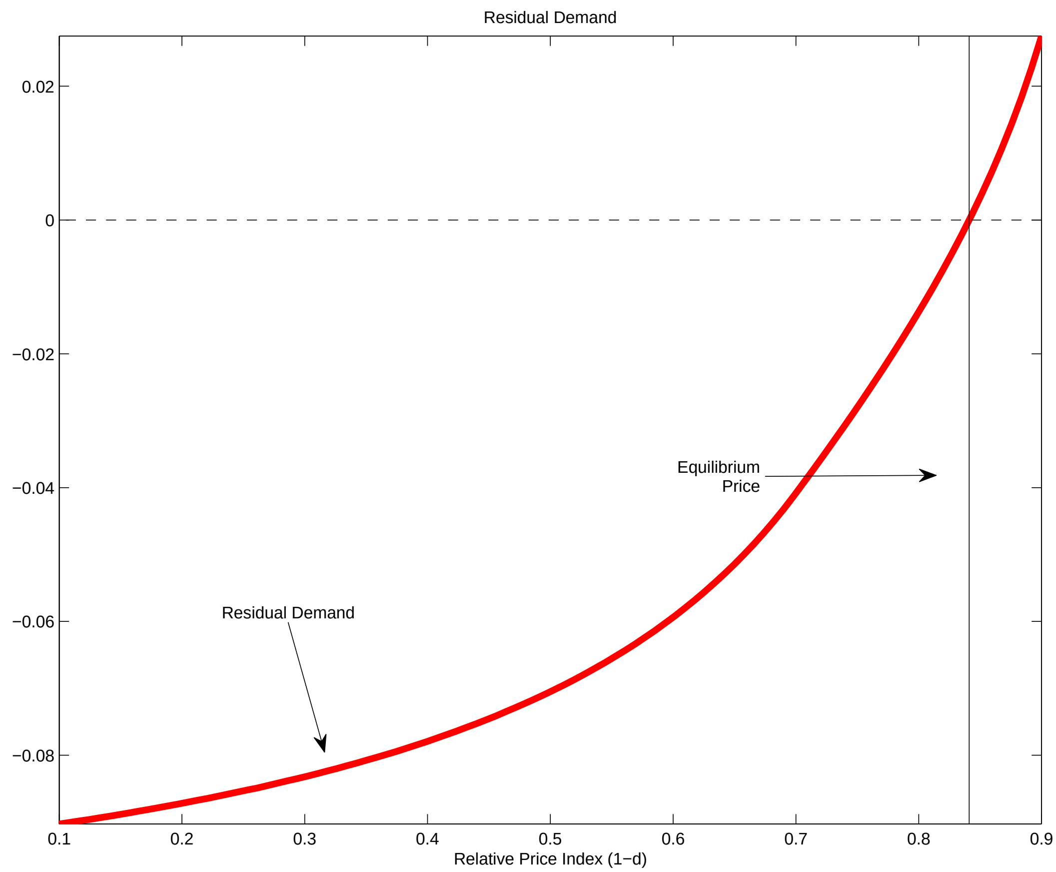

Figure 1: Example of Price Computation

$\varphi = 1$ and $\beta = 0.99$.

Example 2. Now, assume that Π_t is a Beta distribution with parameters μ_a and μ_b and $\pi \in [0, 1]$. Let's approximate the solutions with 100 points.

5 The Continuum Shock Case

In the case of the continuum of shocks, $\Pi_t(\pi^m)$ is a density $\Pi_t(s)$ for $s \in [0, 1]$ and so function the premium:

$$d(X) = \frac{\sum_{m=1}^{n} (1 - \pi^m) \Pi_t(\pi^m) \min((1 - \pi^m), d(X))}{\sum_{m=1}^{n} \Pi_t(\pi^m) \min((1 - \pi^m), d(X))}$$

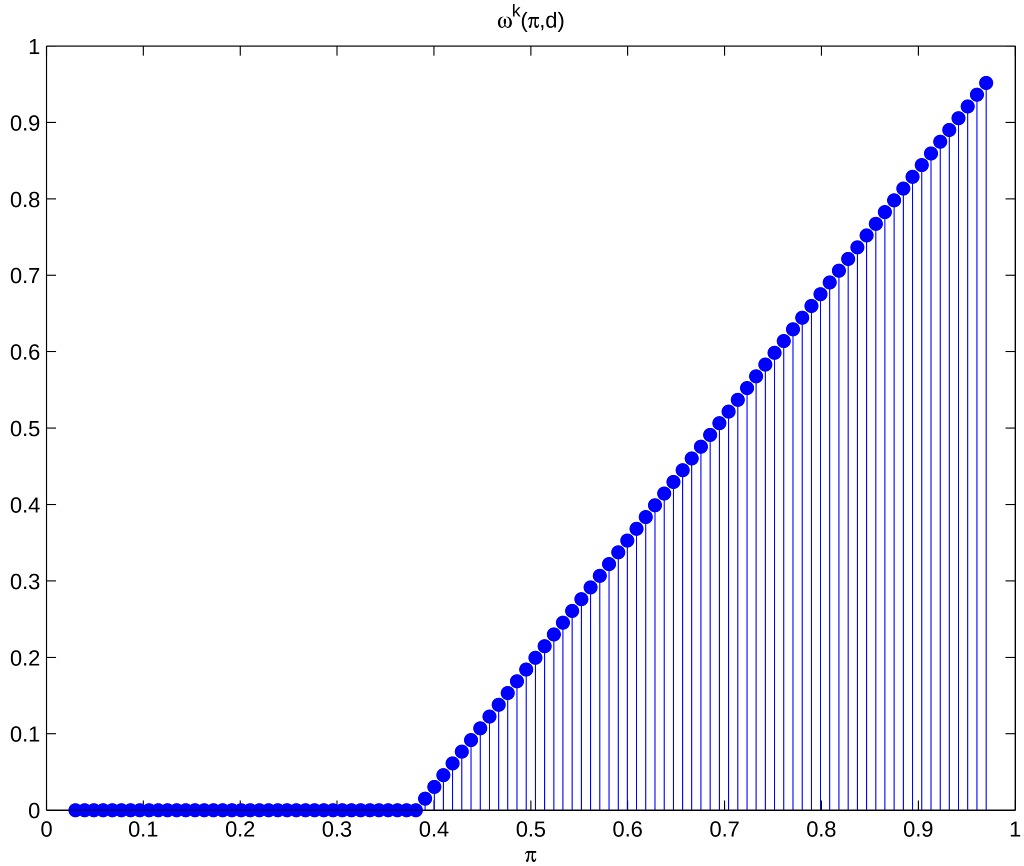

Figure 2: Equilibrium Portfolios

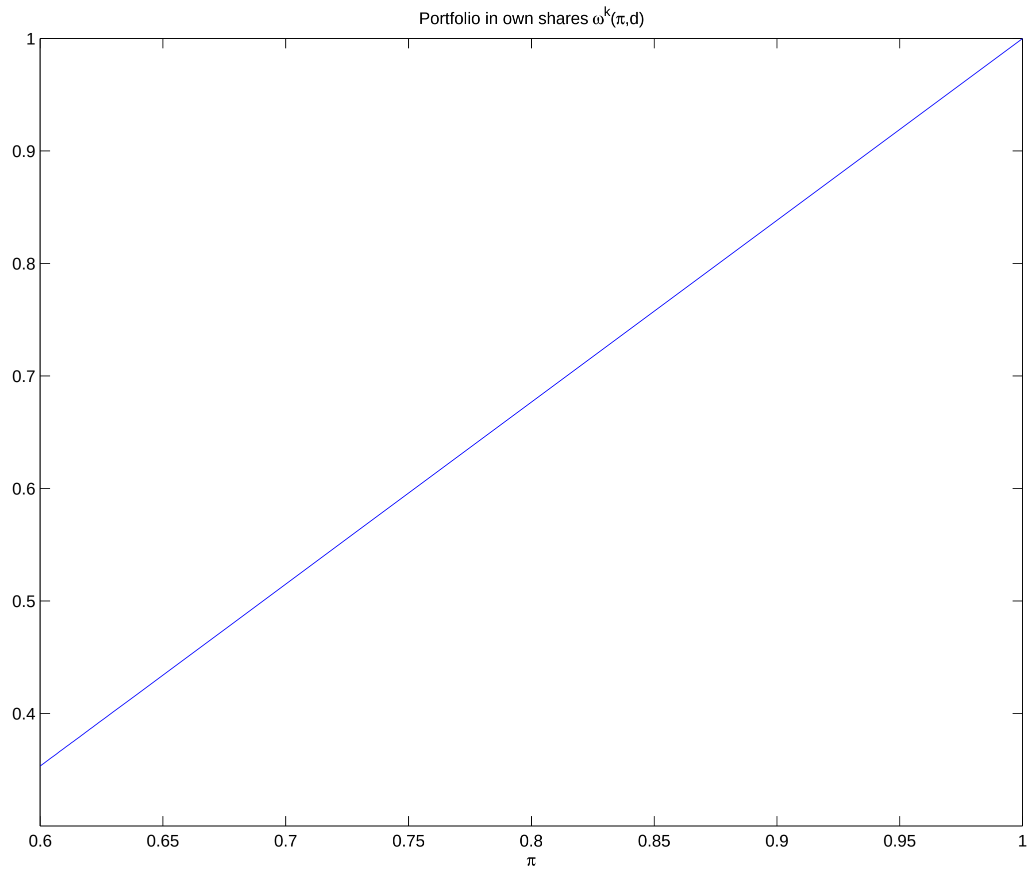

Figure 3: bla bla

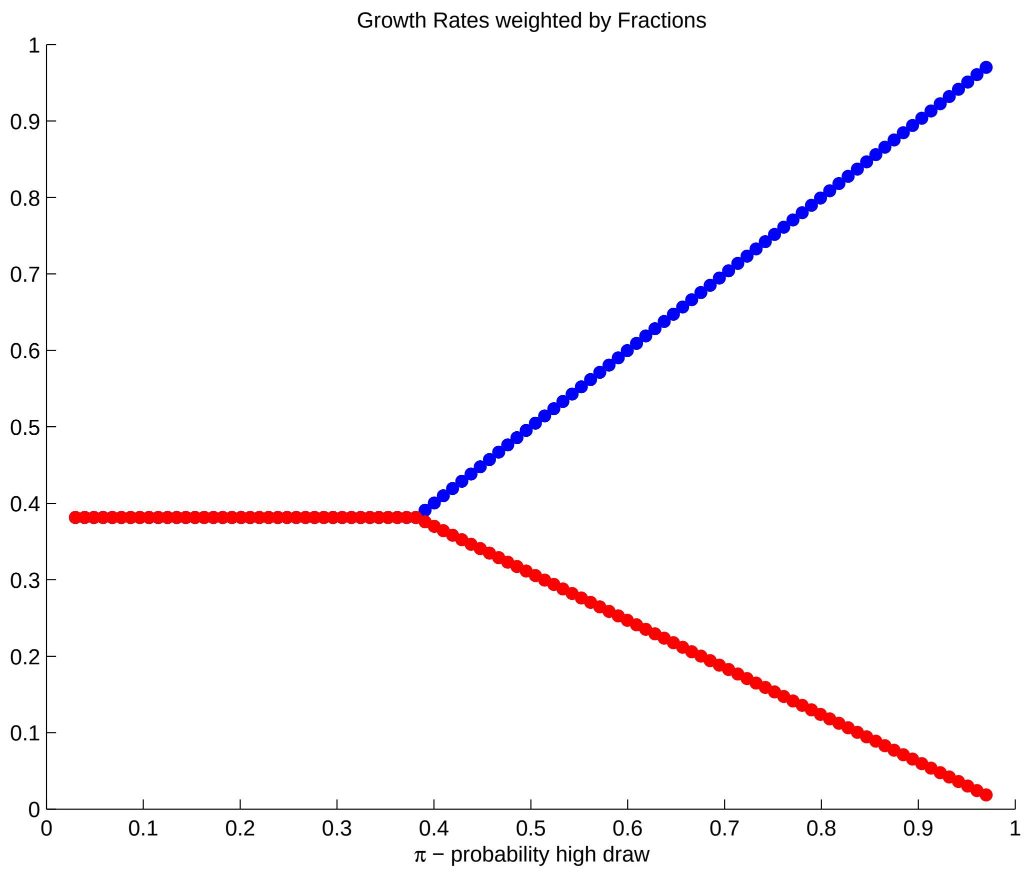

Figure 4: Distribution of Wealth Growth

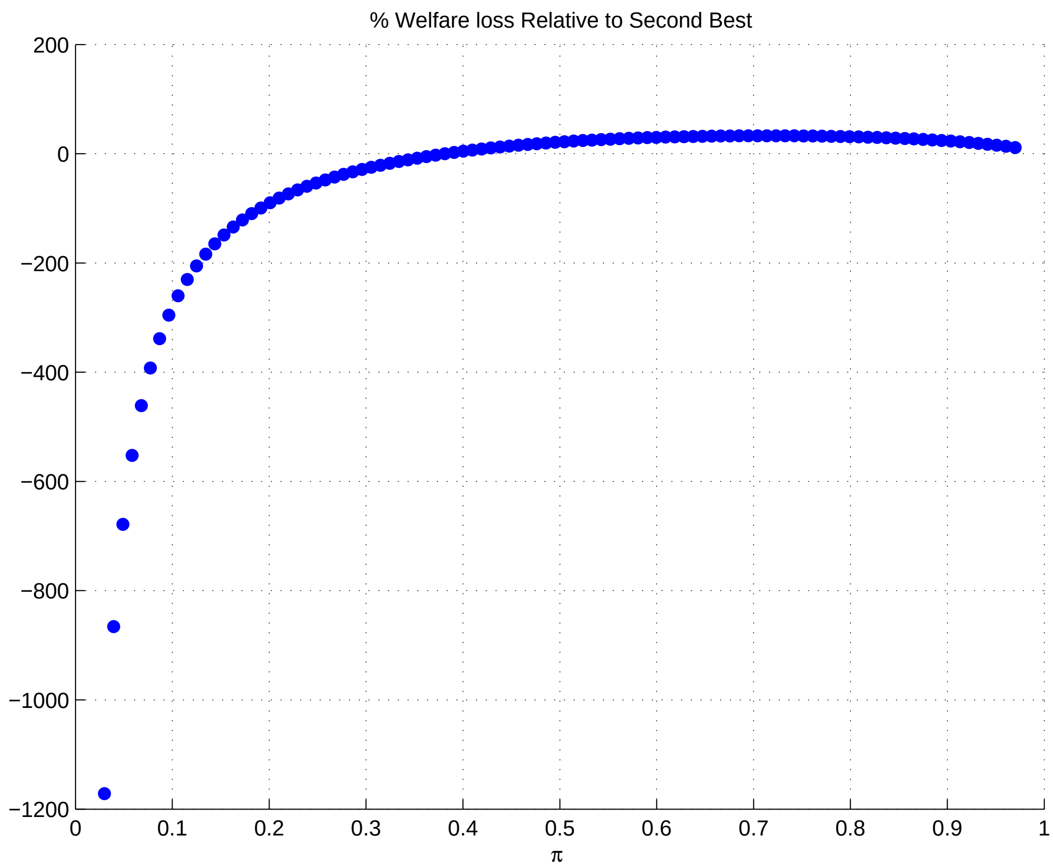

Figure 5: Welfare

is replaced by:

$$d = \frac{\int_0^1 (1-s)\,\Pi_t(s)\min((1-s), d)\,ds}{\int_0^1 \Pi_t(s)\min((1-\pi^m), d)\,ds}$$

$$d = \frac{d\int_0^d (1-s)\,\Pi_t(s)\,ds + \int_d^1 (1-s)^2\,\Pi_t(s)\,ds}{d\int_0^d \Pi_t(s)\,ds + \int_d^1 (1-s)\,\Pi_t(s)\,ds}$$

6 Potential Extensions

- Extend the model to allow preferences different that log, Epstein-Zin to isolate forces
  - Show impact on on aggregate investment, how does this depend on IES parameter?
  - Show that illiquidity bites investment through intertemporal substitution.
  - Explore a feed-back from investment to $\Pi_t \rightarrow$ explore the choice of risk.
- Introduce capital that' doesn't fully depreciate and solve the model.
  - What is the relationship between capital depreciation and the lemons market?
- **Endogenous Choice of Risk.** Suppose that ex-ante, agents can choose the risk of their projects. That is, suppose that they can choose some $\pi(m)$ with $\pi' > 0$ for some index m. Of course, there has to be a cost of chosing m. Assume that φ is also an increasing function of m.
  - Work with a linear function $\pi(m)$ and a convex function $\varphi(m)$. Use paper and pencil as much as possible.
  - Setup a planner's problem.
  - Solve a planner's choice of m when the information structure satisfies the conditions to achieve first-best. Do the same for the second best case. How do the solutions compare? How does the planner's weight on risk-insurance connect with his choice of m?
  - Suppose now that the planner's chooses m, but the competitive equilibrium operates as presented above. What is the planner's choice and how does it compare to the answers to the previous question?
  - Suppose that agents can individually deviate from the choice of m setup by the planner in the previous excercise. Would they choose to do so? Is the solution constrained efficient?
  - Find the solution to the competitive equilibrium that corresponds to the actual choice of m. Identify a pecuniary externality in this model.

7 Environment with intermediary net worth

7.1 Algorithm

The following algorithm can be used to compute equilibria under log preferences.

1. For each $\pi \in \{\pi_1, \pi_2, ..., \pi_n\}$, solve $\Omega(\pi, d)$ for some arbitary $d = \delta$. Call the solutions $\tilde{\omega}^k(\pi, \delta)$.
2. **Would only change here:** $\frac{\sum_{m=1}^{n}(1-\pi^n)\Pi_t(\pi^n)\left(1-\tilde{\omega}^k(\pi^n,\delta)\right)}{\sum_{m=1}^{n}\Pi_t(\pi^n)\left(1-\tilde{\omega}^k(\pi^n,\delta)\right)} = \delta$, where Π we be substituted by the distribution of equity, and π would be location specific.