Fall 2017
Econ 164 Final Exam

Prof. Saki Bigio

December 9th, 2017

The exam has 5 questions for 100 points in total. You have 3 hours to finish the exam.

NOTE THAT POINTS VARY BETWEEN QUESTIONS!

DO NOT OPEN THE EXAM BOOK UNTIL THE EXAM BEGINS!

Name:

UID:

Score:

1. Solow Model (10 points)

Consider the Solow model with population growth and technology progress. Suppose at time 0 the population growth is increased due to an increase in fertility. Answer the following questions.

a. What is the long run effect on capital per capita of an increase in population growth? Show graphically how you reach the conclusion. Explain the intuition behind your answer. (3 points)

b. What is the short run effect on capital per capita? Draw the dynamic path of capital per capita after the increase in population growth rate. (3 points)

c. If we only consider the long run effect, what is effect on GDP per capita (in percent) from an increase of 1% in population [growth rate? level?]? (4 points)

Answers:

a. In this model, capital per effective worker in steady state is

$$\tilde{k}_{ss} = \left(\frac{s}{\delta + n + g}\right)^{1/1-\alpha}$$

Hence, an increase in the population growth rate will decrease the capital per effective worker in steady state. However, this has no impact to the long run growth rate of capital per capita, which grows at the growth rate of technology.

b. There is no jump in the level of capital per capita. Since we have to go to a lower level of capital per efficiency unit, capital accumulation will slow down at first and then accelerate back to the growth rate of technology. Eventually, capital per capita will surpass the levels of K/L before the change in fertility.

c. There is no effect in the long term growth rate, since GDP per capita grows at the growth rate of technology.

2. Good and Bad Government. (20 points)

Consider a version of the Solow model with a Government, which can be either Good or Bad. The law of motion for aggregate capital in the Solow model is given by

$$K_{t+1} = I_t + (1-\delta)K_t.$$

where we assume that investment is a constant proportion of disposable income:

$$I_t = sY_t^D$$

and Y_t^D is disposable income, given by:

$$Y_t^D = Y_t - G_t$$

where we say G_t is given by τY_t.

Finally, assume the total output is given by:

$$Y_t = AK_t^\alpha L_t^{1-\alpha}$$

a. Provide an expression for private investment only as a function of total income. (5 points)

b. Plug in your equation for private investment and write a law of motion for capital per worker. In this case, the government is bad and tosses G_t into the ocean. What is GDP per capita in the steady state? (5 points)

c. Now suppose we have a good government, and that G_t is invested into capital. Provide a new expression for the law of motion for capital, again in terms of Y_t and other parameters of the model. Now what is the steady-state level of GDP per capita? Explain in two sentences how this model describes the important role of institutions. (5 points)

d. Let consumption be given by $C_t = (1-s)Y_t^D$. Suppose a good government can't control private savings (s) but it can control its own level of taxing (τ) - what is the best choice of τ? Provide an intuition for this rule. (5 points)

Answers:

a. As announced in class, the question should say PRIVATE investment, not total investment. $I^{prv} = sY_t^D = s(Y_t - G_t) = s(Yt - \tau Y_t) = s(1-\tau)Y_t$. Total investment would just be this plus government investment, but we do not yet know what the government chooses for I_t^G.

b. Any assumption about growth of technology or labor is fine. Here we allow for them to grow, but most students on the exam (as announced in class) assumed no growth in technology or labor.

$$
\begin{aligned}
K_{t+1} &= I_t^{prv} + I_t^G + (1-\delta)K_t \\
K_{t+1} &= s(1-\tau)Y_t + 0 + (1-\delta)K_t \\
(1+\tilde{x})(1+n)\frac{K_{t+1}}{\tilde{A}_{t+1}L_{t+1}} &= s(1-\tau)\frac{Y_t}{\tilde{A}_t L_t} + (1-\delta)\frac{K_t}{\tilde{A}_t L_t} \\
(1+\tilde{x})(1+n)k_{t+1} &= s(1-\tau)y_t + (1-\delta)k_t
\end{aligned}
$$

In the steady state, we know that $k_{t+1} = k_t = k_{ss}$ so we have:

$$
\begin{aligned}
(1+\tilde{x})(1+n)k_{ss} &= s(1-\tau)y_{ss} + (1-\delta)k_{ss} \\
\frac{(\delta+\tilde{x}+n+n\tilde{x})}{s(1-\tau)}k_{ss} &= y_{ss}
\end{aligned}
$$

This equation is correct but it doesn't fully answer the question since we know that y_{ss} and k_{ss} are (potentially) functions of each other. Thus, we now make the standard assumption from the Solow model that $y_{ss} = k_{ss}^{\alpha}$ in order to keep making progress, giving us:

$$
\begin{aligned}
\frac{(\delta+\tilde{x}+n+n\tilde{x})}{s(1-\tau)}k_{ss} &= k_{ss}^{\alpha} \\
k_{ss} &= \left(\frac{s(1-\tau)}{\delta+\tilde{x}+n+n\tilde{x}}\right)^{\frac{1}{1-\alpha}} \\
\frac{Y_{t,bg}}{L_{t,bg}} &= \tilde{A}_{t,bg}\left(\frac{s(1-\tau)}{\delta+\tilde{x}+n+n\tilde{x}}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

c. This question is almost the same as the previous question, except now we have $I_t^G = \tau Y_t$ instead of being zero. This gives us:

$$
\begin{aligned}
K_{t+1} &= I_t^{prv} + I_t^G + (1-\delta)K_t \\
K_{t+1} &= s(1-\tau)Y_t + \tau Y_t + (1-\delta)K_t \\
K_{t+1} &= [s+\tau(1-s)]Y_t + (1-\delta)K_t
\end{aligned}
$$

From here, the steps follow exactly as they did in the answer to part (b) except we have a different constant in front of Y_t. The end result is that we get:

$$\begin{aligned} k_{ss} &= \left(\frac{s+\tau(1-s)}{\delta+\tilde{x}+n+n\tilde{x}}\right)^{\frac{1}{1-\alpha}} \\ \frac{Y_{t,bg}}{L_{t,bg}} &= \tilde{A}_{t,bg}\left(\frac{s+\tau(1-s)}{\delta+\tilde{x}+n+n\tilde{x}}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Any positive level of taxation will reduce GDP per capita, but the government can partially offset the negative effects by reinvesting the tax revenue into new capital. Therefore institutions have the power to change the shape of a country's economy.

d. Private consumption will be given by $C_t = (1-s)Y_t^D = (1-s)(1-\tau)Y_t$. Plugging in our result from part (c) this gives us:

$$\begin{aligned} \frac{C_{t,bg}}{L_{t,bg}} &= (1-s)(1-\tau)\tilde{A}_{t,bg}\left(\frac{s+\tau(1-s)}{\delta+\tilde{x}+n+n\tilde{x}}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \frac{(1-s)\tilde{A}_{t,bg}}{(\delta+\tilde{x}+n+n\tilde{x})^{\frac{\alpha}{1-\alpha}}}(1-\tau)[s+\tau(1-s)]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

The question of the optimal choice of τ is then just asking what τ maximizes the value of the above expression. Since everything before $(1-\tau)$ is a constant with respect to τ we ignore it in the calculation below. We proceed by taking the derivative with respect to τ and setting it equal to zero.

$$
\begin{aligned}
0 &= \frac{d}{d\tau}(1-\tau)[s+\tau(1-s)]^{\frac{\alpha}{1-\alpha}} \\
0 &= -[s+\tau(1-s)]^{\frac{\alpha}{1-\alpha}} + (1-\tau)\frac{\alpha}{1-\alpha}[s+\tau(1-s)]^{\frac{\alpha}{1-\alpha}-1}(1-s) \\
0 &= -1 + (1-\tau)\frac{\alpha}{1-\alpha}[s+\tau(1-s)]^{-1}(1-s) \\
1 &= \frac{\alpha}{1-\alpha}(1-\tau)[s+\tau(1-s)]^{-1}(1-s) \\
[s+\tau(1-s)] &= \frac{\alpha}{1-\alpha}(1-\tau)(1-s) \\
\tau(1-s) &= \frac{\alpha}{1-\alpha}(1-s) - \frac{\alpha}{1-\alpha}(1-s)\tau - s \\
\left(1+\frac{\alpha}{1-\alpha}\right)(1-s) &= \frac{\alpha}{1-\alpha}(1-s) - s \\
(1-\alpha+\alpha)(1-s)\tau &= \alpha(1-s) - s(1-\alpha) \\
(1-s)\tau &= \alpha - s\alpha - s + s\alpha \\
(1-s)\tau &= \alpha - s \\
\tau &= \frac{\alpha - s}{1-s}
\end{aligned}
$$

Let's consider if this expression makes sense. We know from class that when $s = \alpha$ consumption is maximized. In this case, when $s = \alpha$ then $\tau = 0$, essentially saying that if the private sector is already behaving optimally there is no room for government to improve. On the other hand, if $s = 0$ then all savings in the economy is done by the government and $\tau = \alpha$. Other than these extreme cases, though, is using τ better or worse than controlling s? Using our result for τ we have:

$$
\begin{aligned}
(1-s)(1-\tau)[s+\tau(1-s)]^{\frac{\alpha}{1-\alpha}} &= (1-s)\left(1-\frac{\alpha-s}{1-s}\right)\left[s+\frac{\alpha-s}{1-s}(1-s)\right]^{\frac{\alpha}{1-\alpha}} \\
&= (1-s-\alpha+s)[s+\alpha-s]^{\frac{\alpha}{1-\alpha}} \\
&= (1-\alpha)[\alpha]^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

This is the same result we had in class when we had only total savings $s = \alpha$ and no government sector. The intuition is that government can use taxation to achieve the optimal level (i.e. the one that maximizes consumption) of aggregate savings.

3. Solow Model and transition dynamics (15 points)

We have learned that capital per efficiency units of labor in the Solow model is given by the following equation:

$$\tilde{k}_t = \left[\frac{\mathbf{s}}{\delta + g} + \left(\tilde{k}_0^{1-\alpha} - \frac{\mathbf{s}}{\delta + g}\right) \exp\left(-\left(1-\alpha\right)\left(\delta + g\right)t\right)\right]^{\frac{1}{1-\alpha}}.$$

Answer the following questions using this equation. Use the following standard numbers for the Solow Model when necessary $s = 0.25$, $\alpha = 0.30, g = 0.015$ and $\delta = 0.05$.

a. From this equation, derive the steady state $\tilde{k}_{ss}$ for capital per efficiency units of labor. (5 points)

For questions b and c, suppose that a country has suffered a hurricane at $t = 0$. During the hurricane, 25% of the capital stock was destroyed.

b. Derive an expression (in terms of the parameters of the model) for the time it will take for $\tilde{k}_t$ to be 0.1% below steady state. (7 points)

c. Find an exact numeric value for the time needed for k_t to reach 0.1% below state state? (3 points)

a. From last equation,

$$\lim_{t\to\infty} \tilde{k}_t^{1-\alpha} = \frac{\mathbf{s}}{\delta + g} + \lim_{t\to\infty} \left(\tilde{k}_0^{1-\alpha} - \frac{\mathbf{s}}{\delta + g}\right) \exp\left(-\left(1-\alpha\right)\left(\delta + g\right)t\right) \tag{1}$$

Assuming $\tilde{k}_0 > 0$,

$$\lim_{t\to\infty} \left(\tilde{k}_0^{1-\alpha} - \frac{\mathbf{s}}{\delta + g}\right) \exp\left(-\left(1-\alpha\right)\left(\delta + g\right)t\right) = 0$$

Hence

$$\lim_{t\to\infty} \tilde{k}_t = \left(\frac{\mathbf{s}}{\delta + g}\right)^{\frac{1}{1-\alpha}}$$

b. We want $\tilde{k}_t = 0.999\tilde{k}_{ss}$ and we have $k_0 = 0.75k_{ss}$, therefore

$$0.999^{1-\alpha}\left(\frac{s}{\delta + g}\right) = \frac{\mathbf{s}}{\delta + g} + \left(0.75^{1-\alpha}\left(\frac{s}{\delta + g}\right) - \frac{\mathbf{s}}{\delta + g}\right) \exp\left(-\left(1-\alpha\right)\left(\delta + g\right)t\right)$$

Which simplifies to

$$0.999^{1-\alpha} - 1 = \left(0.75^{1-\alpha} - 1\right) \exp\left(-\left(1-\alpha\right)\left(\delta + g\right)t\right)$$

Divide and take the log

$$\ln\left(\frac{0.999^{1-\alpha}-1}{0.75^{1-\alpha}-1}\right)=-\left(1-\alpha\right)\left(\delta+g\right)t$$

Thus

$$t=-\frac{1}{\left(1-\alpha\right)\left(\delta+g\right)}\ln\left(\frac{0.999^{1-\alpha}-1}{0.75^{1-\alpha}-1}\right)$$

c. Replace the numbers in the previous formula to get $t\simeq 122.257$

4. Misallocation (25 points)

Consider the following model of misallocation of inputs. There are two sectors in this economy, $s = 1, 2$. Both sectors produce using labor according to the following production function

$$y_s = A_s^{1-\alpha} l_s^{\alpha}$$

Where A_s is a sector specific technological level and l_s is the demand for labor inputs from sector s. We assume $\alpha \in (0, 1)$. The output of both sectors are equally valuable for consumers. Thus, total output is the sum of output of the two sectors:

$$Y = y_1 + y_2.$$

The resource constraint of this economy is given by $l_1 + l_2 = L$. Here L is the population of this economy, which we assumed to be fixed.

1. Compute the efficient allocation in this model. How does the allocation of labor depends on sectoral productivity? Find an expression for the efficient total output, Y. (3 points)

2. Let w_t be the wage per hour paid by both firms. Define the competitive equilibrium in this economy. (1 points)

3. Now suppose that the government of this country concedes a subsidy to production in sector 1. The subsidy is such that the revenue received by firms in sector 1 is equal to $(1+\sigma)y_1$, where $\sigma > 0$. Find an equation that characterizes the optimal labor demand of sector 1. To do so, define the profit maximization problem for a firm in sector 1. (3 points)

4. To achieve fiscal budget balance, the government taxes firms in sector 2. The tax is such that the revenue perceived firms in sector 2 is equal to $(1 - \tau)y_2$, where $\tau > 0$. Find an equation that characterizes the optimal labor demand of sector 2. (3 points)

5. Using your previous results and the equilibrium in the labor market, derive an expression for the total GDP in this economy in terms of α, L, τ and σ. Explain how this model captures the effects of taxes on misallocation. Explain how Government distortions affect TFP in the context of this model. (10 points)

6. Find an expression in terms of the model parameters for the tax τ required in this model to achieve fiscal budget balance. (5 points)

Answers:

1. Compute the efficient allocation in this model. How does the allocation of labor depends on sectoral productivity? Find an expression for the efficient total output, Y.

   In the optimal allocation:
$$\frac{A_1}{A_2} = \frac{l_1}{l_2}$$
   Hence
$$\frac{A_1}{A_2} = \frac{L - l_2}{l_2}$$
   So
$$l_1 = \frac{A_1}{A_1 + A_2} L \tag{2}$$
$$l_2 = \frac{A_2}{A_1 + A_2} L \tag{3}$$
   Hence, labor in each sector is allocated proportionally to total labor, with weights depending on productivities. Hence,
$$Y = (A_1 + A_2)^{1-\alpha} L^{\alpha} \tag{4}$$

2. Let w_t be the wage per hour paid by both firms. Define the competitive equilibrium in this economy.

   A competitive equilibrium is a set of :

   - labor demands l_1 and l_2
   - Output in sector 1 and 2, y_1 and y_2
   - Wages w

   such that

   - Labor market clears, $l_1 + l_2 = L$
   - Firms maximize their profits

   In the competitive equilibrium, $w = \alpha \left(\frac{A_1+A_2}{L}\right)^{1-\alpha}$ with $l_2 = \frac{A_2}{A_1+A_2} L$, $l_1 = \frac{A_1}{A_1+A_2} L$.

3. Now suppose that the government of this country concedes a subsidy to production in sector 1.

   Firms in sector 1 maximize

$$\pi_1 = (1+\sigma)A_1^{1-\alpha} l_1^{\alpha} - w l_1$$

   Hence the optimal demand for labor in this sector is

$$l_1 = A_1 \left(\frac{\alpha\,(1+\sigma)}{w} \right)^{\frac{1}{1-\alpha}} \tag{5}$$

   Note that given w, l_1 increases in both A_1 and σ.

4. To achieve fiscal budget balance, the government taxes firms in sector 2. The tax is such that the revenue perceived firms in sector 2 is equal to $(1-\tau)y_2$, where $\tau > 0$. Find an equation that characterizes the optimal labor demand of sector 2.

   Firms in sector 2 maximize

$$\pi_2 = (1-\tau)A_2^{1-\alpha} l_2^{\alpha} - w l_2$$

   Hence the optimal demand for labor in this sector is

$$l_2 = A_2 \left(\frac{\alpha\,(1-\tau)}{w} \right)^{\frac{1}{1-\alpha}} \tag{6}$$

5. Using your previous results and the equilibrium in the labor market, find an expression for the total GDP in this economy in terms of α, L, τ and σ. Explain how does this model capture the effects of taxes on misallocation.

   We have

$$\frac{l_1}{l_2} = \frac{A_1}{A_2} \left(\frac{1+\sigma}{1-\tau} \right)^{\frac{1}{1-\alpha}}$$

   Denote $\mu = \left(\frac{1+\sigma}{1-\tau}\right)^{\frac{1}{1-\alpha}}$. Hence

$$\frac{l_1}{l_2} = \frac{A_1}{A_2}\mu$$

   so μ can be interpreted as a wedge that erodes efficiency in this economy. Using the labor market equilibrium

$$\frac{L-l_2}{l_2} = \frac{A_1}{A_2}\mu$$

Hence

$$l_2 = \frac{A_2}{A_1\mu + A_2} L \tag{7}$$

and

$$l_1 = \frac{A_1\mu}{A_1\mu + A_2} L \tag{8}$$

We can use our results and get

$$Y = \left\{ A_1^{1-\alpha} \left(\frac{A_1\mu}{A_1\mu + A_2} \right)^{\alpha} + A_2^{1-\alpha} \left(\frac{A_2}{A_1\mu + A_2} \right)^{\alpha} \right\} L$$

Going back to our previous notation

$$Y = \left\{ \frac{A_1(1+\sigma)^{\frac{\alpha}{1-\alpha}} + A_2(1-\tau)^{\frac{\alpha}{1-\alpha}}}{\left(A_1(1+\sigma)^{\frac{1}{1-\alpha}} + A_2(1-\tau)^{\frac{1}{1-\alpha}} \right)^{\alpha}} \right\} L^{\alpha}$$

So the introduction of taxes affects the marginal productivity of labor in each sector and hence TFP.

6. Find an expression in terms of the model parameters for the tax τ required in this model to achieve fiscal budget balance.

   To achieve fiscal budget balance

$$\underbrace{\tau y_2}_{\text{Revenue}} = \underbrace{\sigma y_1}_{\text{Expenditure}}$$

Hence

$$\tau A_2^{1-\alpha} \left(\frac{A_2}{A_1\mu + A_2} L \right)^{\alpha} = \sigma A_1^{1-\alpha} \left(\frac{A_1\mu}{A_1\mu + A_2} L \right)^{\alpha}$$

Hence

$$\tau A_2 = \sigma A_1 \mu^{\alpha}$$

So

$$\tau = \frac{A_1}{A_2} \sigma \left(\frac{1+\sigma}{1-\tau} \right)^{\frac{\alpha}{1-\alpha}} \tag{9}$$

5. Schumpeterian Model (30 points)

Consider the Schumpeterian model with one-period patent. Output is produced according to:

$$Y_t = A_t L_t,$$

where L_t is number of production workers and A_t describes the technology available to some or many firms. The number of production workers is fixed at $\bar{L}$.

At period $t-1$, all firms have access to the same technology A_{t-1}. One innovating firm invests in $R\&D$ to achieve a better technology. If $R\&D$ is successful, the innovating firm becomes a leader next period with $A_t = (1+\gamma)A_{t-1}$ while the other firms still have the now inferior technology A_{t-1}. The patent only holds for one period, after that all firms have access to the new technology. If $R\&D$ is not successful, all firms have the old technology.

The probability of successful $R\&D$ depends on the amount of research workers employed,

$$\pi\left(n_{t-1}\right) = \nu \frac{n_{t-1}^{1-\sigma}}{1-\sigma}.$$

where π is the probability of success, and n_{t-1} is the number of research workers employed. The research workers are paid the same wage as the production workers.

Markets are competitive. Firms maximize profits in production and $R\&D$. They discount future profit at rate β. Answer the following questions. (30 points)

a. If $R\&D$ is successful, what is the leaders profit? (5 points)

b. Using your previous results from question a), please write down the expected profits from $R\&D$. Derive the demand for research workers. (5 points)

c. Derive the social value of $R\&D$ given a number of workers in $R\&D$, n_{t-1}. This social planner takes into account the probability of further innovation and the cost of innovation. From the perspective of the social planner, the probability of further innovation in any period is $\tilde{\pi}$. (5 points)

d. To maximize the social value of $R\&D$, how many workers should be employed? (5 points)

e. Suppose that $v = 1/2$, $\sigma = 1/2$, $\gamma = 0.1$, $\bar{L} = 1$, $\tilde{\pi} = 0.01$, $\beta = 0.98$ and $A_{t-1} = 1$. Using these parameters, compute the social optimal demand for labor in $R\&D$. Please compute the market $R\&D$ employment. Does the level of $R\&D$ employment chosen by the firm fall behind or above the optimal one? Explain the intuition. (10 points)

Answers:

a. First we will consider the costs of innovation:

- This firm hires workers, n_0.

- The pay per hour is w_0. Here $w_0 = A_0$
- Cost of innovation $= n_0 A_0$, and it is paid at time zero regardless of the success of innovation.

If there is innovation, this firm will have technology $A_0(1+\gamma)$. In $t = 1$.

- They will pay w_1 per hour worked in the production sector.
  - What is the equilibrium w_1?
    * The followers will pay at most their marginal productivity, A_0
    * If the leader pays workers A_0, the workers would be willing to move.
    * There are no incentives to pay them more, so $w_1 = A_0$

Then we can compute profits in this period

$$\underbrace{A_0(1+\gamma)\bar{L}}_{\text{Revenue}} - \underbrace{A_0\bar{L}}_{\text{Cost}} = \gamma A_0 \bar{L}$$

b.

$$E(Profits) = \Pr(\text{Innovation}) \times \underbrace{\beta\gamma A_0\bar{L}}_{\substack{\text{discounted}\\ \text{profitsfrom}\\ t=1}} - \underbrace{A_0 n_0}_{\substack{\text{Costs}\\ \text{of}\\ \text{Innovation}}}$$

Assume that

$$\Pr(\text{Innovation}) = v\frac{n_0^{1-\theta}}{1-\theta}$$

A higher n_0 implies a higher prob of innovation. Parameter v controls that the probability sums 1. Here θ is the elasticity of the prob of innovation: How responsive is the marginal probability to changes in n_0.

From expected profits we see that there is a trade off:

- A high n_0 implies that the firm is more likely to have success
- A low n_0 costs less

The optimal allocation implies

$$\underbrace{v\, n_0^{-\theta}\left(\beta\gamma A_0\bar{L}\right)}_{\text{Marginal expected benefits}} = \underbrace{A_0}_{\text{Marginal Cost}}$$

This determines n_0

$$n_0^\star = \left(v\beta\gamma\bar{L}\right)^{\frac{1}{\theta}} \tag{10}$$

c) Remember that the gain of innovation after the patent expires in any period is $\gamma A_0\bar{L}$. So

$$E(Gain) = \{\Pr(\text{innovation}) \sum_{i=1}^{\infty} \underset{\text{discount}}{\underset{\downarrow}{\beta^i}} \times \Pr(\text{no further innovation until } i) \times \underbrace{\gamma A_0 \bar{L}}_{\text{gain in wages}} -$$

Now, the probability of further innovation:

$$\Pr(\text{no further innovation until } i) = \underset{\text{No Innov. in } t=2}{(1-\tilde{\pi})} \quad \underset{\text{No Innov. in } t=3}{(1-\tilde{\pi})} \quad \cdots \quad \underset{\text{No Innov. in } t=i}{(1-\tilde{\pi})}$$

So

$$\Pr(\text{no further innovation until } i) = \big(1 - \underset{\text{Prob of innovation}}{\underset{\downarrow}{\tilde{\pi}}} \underset{\text{No further Innovation}}{\underset{\downarrow}{\big)^{i-1}}}$$

For the first part note that

$$\sum_{i=1}^{\infty} \beta^i \times \Pr(\text{no further innovation until } i) \times \gamma A_0 \bar{L} = \beta \sum_{i=0}^{\infty} \beta^i (1-\bar{\pi})^i \gamma A_0 \bar{L}$$

Then

$$\sum_{i=1}^{\infty} \beta^i (1-\bar{\pi})^i \gamma A_0 \bar{L} = \frac{\beta\gamma A_0 \bar{L}}{1-\beta(1-\tilde{\pi})}$$

Hence

$$E(Gain) = v\frac{n_0^{1-\theta}}{1-\theta}\,\frac{\beta}{1-\beta(1-\tilde{\pi})}\gamma A_0 \bar{L} - A_0 n_0$$

d. So what would do the social planner? Taking derivatives:

$$vn_0^{-\theta}\frac{\beta}{1-\beta(1-\tilde{\pi})}\gamma\bar{L}A_0 - A_0 = 0 \tag{11}$$

So

$$n_0^{SP} = \left(v\frac{\beta}{1-\beta(1-\tilde{\pi})}\gamma\bar{L}\right)^{\frac{1}{\theta}} \tag{12}$$

e. Now we use our previous results. Remember that the allocation that the social planner would chose is

$$n_0^{SP} = \left(v\frac{\beta}{1-\beta(1-\tilde{\pi})}\gamma\bar{L}\right)^{\frac{1}{\theta}}$$

While the market allocation is

$$n_0^{\star} = \left(v\beta\gamma\bar{L}\right)^{\frac{1}{\theta}}$$

So in general, for any $\tilde{\pi} < 1$

$$\frac{\beta}{1-\beta(1-\tilde{\pi})} > \beta$$

Since θ is positive, $f(x) = x^{1/\theta}$ is increasing, so

$$n_0^{SP} > n_0^{\star}$$

We are asked for the allocations using this set of parameters so

$$n_0^{\star} = \left(0.98 * 0.1 * \frac{1}{2}\right)^2 = 0.00241$$

$$n_0^{s} = \left(\frac{0.98 * 0.1 * \frac{1}{2}}{1-0.98*(1-0.01)}\right)^2 = 2.703707$$