Lecture Notes: Financial Intermediation
Brunnermeier Sannikov 2011

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1 Overview

These notes review three differnet papers on financial interemedation. These are, Brunnermeier-Sannikov 2011, He and Krishnamurthy 2012, and Adrian and Boyarchenko 2012. I present the papers in their discrete time formulation. Ther are some common featuers in all three of these papers:

- Like in any model with intermediaries, intermediaries are essential for the allocation of resources.
- The game is that constraints on intermediaries, will affect allocations of resources to technlogies they manage.
- This will have effects on output, and the composition of investment and consumption.
- The three models differ in some details whereby intermediaries net-worth relaxes financial constraints.
  - In equilibrium, intermediaries have higher exposure to financial risk.
  - In addition, risk is augmented by shocks, but also through feed back effects through prices.
    - There are no fire-sale effects.
    - Negative shocks affect prices because capital becomes less valuable if intermediaries are hit.
    - Essentially, when intermediaries are hit, they provide less intermediation, but this lowers the return to capital because they are more productive in their use.

2 Environment

Demographics. The economy is populated by a continuum of households and a continuum of intermediaries. Intermediaries and households can live (stochastic death).

Preferences. Both intermediaries and households have homothetic utility (possibly linear). Let's denote the identity of each group by $\{h, i\}$. The corresponding discount rate is $\beta^j, j \in \{h, i\}$. In general, we assume $\beta^h < \beta^i$.

They evaluate consumption streams accordingly via:

$$\max \sum_{t \geq 0}^{\infty} \left[\left(\beta^j\right)^t U^j \left(c_t^j\right) \right] \text{ for } j \in \{h, i\}$$

In addition, households are allowed to have negative consumption (interpreted as labor) but intermediaries are restricted to have positive consumption. Technologies are linear.

Shocks and Timing. There's a unique aggregate shock ϕ_t that takes discrete values. The shock ϕ_t determines the efficiency units that remain from a unit of capital once shocks hit, through an agent-specific depreciation function

$\lambda^j(\phi_t)$ and also affects the productivity $A^i(\phi_t)$. The shock arrives after capital has been allocated among agents. All decisions are made before the arrival of the shock.

Technology. Both intermediaries and households can hold capital. There are two discitions though, first, intermediaries are more productive and second, for any state ϕ_t. Second, intermediaries have access to an investment technology.

Production is carried out according to linear technologies:

$$y_t^j = A_t^j k_t^j \text{ for } j \in \{h, i\}$$

where k_j is the capital stock of an agent j. By assumption, $A^h(\phi_t) < A^i(\phi_t)$. The evolution of the physical units managed by households is:

$$k_{t+1}^h = \lambda(\phi) k_t^h$$

and for intermediaries is

$$k_{t+1}^i = \Phi(\iota_t) k_t^i + \lambda(\phi) k_t^i$$

where ι_t is the intermediaries investment rate and Φ reflects an investment cost function, with the property that $\Phi(0) = 0$, $\Phi' > 0, \Phi'' < 0$.

Adjustement costs must be done per efficiency unit.

Markets. There are incomplete markets. In particular, there's only one debt asset. These debt asset can be issued by both agents and is long-term debt. An unit is sold at price q_t and entitles its holders to a stream of consumption $(1-\delta)\delta^{s-1}$ is period s periods after the issuance. When $\delta = 1$, this is the standard one period debt. It is a convenient way to introduce long-term debt in a tractable way. If an agent holds a stock b_t of such debt.

Note that in period (s+t), the payment on debt equals $(1-\delta)\delta^{s-1}$. Now if the same stock of debt is issued in period (t+s-1), the payment is equal to $\left((1-\delta)\delta^{s-1}\right)\delta$, which is the same payment of debt issued in period t, if one depreciates the ammount by δ. This shows that this form of debt

$$b_t = l_t + \delta b_t$$

where l_t is the issuance of debt during period t.

In addition to debt markets, there is a market for physical capital. The price of capital is p_t.

Accounting. Every period, there's an aggregate stock of capital K_t of which the fraction

$$\psi_t = \frac{\int k_t^j di}{K_t}$$

is held by intermediaries and ther rest, $(1-\psi_t)$, held by households. Now, since capital run by different agents has a different return, capital cannot be shorted so k_t^j. It is important to note that there's no rental market for capital. If there were we would need to introduce similar frictions to the ones introduced so far. Accounting identities will verify that at every point in time:

$$N_t = \int n_t^i di + \int n_t^h dh$$

The market clearing condition for the debt market in this economy is given by:

$$\int b_t^i di + \int b_t^h dh = 0.$$

Now, the net worth of an agent is given by:

$$n_t^j = p_t k_t^j + q_t b_t^j.$$

Hence, integrating accros both guys yields the total value of capital in a given point in time is $N_t = q_t K_t$. Intermediaries and households are characterized by state variables n_t^i and n_t^h which correspond to their net-worths.

Finally, the goods market clearing condition requires:

$$\int c_t^i di + \int c_t^h dh = \int \left[A^i - \iota_t^i\right] k_t^i di + \int A^h k_t^h dh$$

The evolution of aggregate capital follows:

$$K_{t+1} = \int \left[\Phi\left(\iota_t^i\right) + \lambda\left(\phi\right)\right] k_t^i di + \int \lambda\left(\phi\right) k_t^h dh$$

and accounting yields:

$$\int k_t^i di + \int k_t^h dh = K_t.$$

By Walras's law, there's no need to check for the bond market clearing.

They key characteristic of this environment is that the equilibrium depends on the way in which n_t^i will evolve. In particular, it will be the case that $n_t^i \leq p\left(X\right) k_t^i$, meaning that intermediaries will manage more than the capital that they could possibly purchase. However, in all three papers, they will be contrained by some variable:

$$b_t^i \leq \theta_t n_t^i.$$

The three papers share very similar structure, however, they differ in the nature of the constraint. We will see that this constraint will play a very important role in the dynamics.

2.1 Agent Problems

Let me use a markovian representation. Let X be the relevant state determining prices. Later we show that this representation exists. Since both problems related, I present them togther:

$$\begin{array}{cc}
\underline{\text{Household's}} & \underline{\text{Intermediaries's}} \\
V^h\left(k,b;X\right) = \max_{k',b',c} U^h\left(c\right) + \beta^h \mathbb{E}\left[V^h\left(k,b;X\right)\right] & V^i\left(k,b;X\right) = \max_{i,k',b',c} U^i\left(c\right) + \beta^i \mathbb{E}\left[V^i\left(k,b;X\right)\right] \\
\text{subject to} & \text{subject to} \\
c + p\left(X\right) k' + q\left(X\right) b' = W^h\left(k,b;X\right) & c + p\left(X\right) k' + q\left(X\right) b' = W^i\left(k,b;X\right) \\
W^h\left(k,b;X\right) = \left(A^i\left(X\right) + q\left(X\right)\lambda^i\left(X\right)\right) k + \left(\left(1-\delta\right) + \delta p\left(X\right) b\right) & W^i\left(k,b;X\right) = \left(A\left(X\right) - i + q\left(X\right)\lambda\left(X\right)\right) k + q\left(X\right)\Phi\left(i\right) + \left(\left(1\right.\right. \\
k \geq 0 & k \geq 0;\ b^i\left(X\right) \leq \theta\left(X\right) n^i\left(X\right)
\end{array}$$

so the only difference between intermediaries and households, aside from the technology is the constraint.

There are some important things to note. The role of four objects.

1. The convex adjustment costs, Φ are very important to deliver variation in prices. Otherwise, the price of capital would be one. There wouldn't be much action on this dimension.

2. Technology and preferences (stochastic death) have to be different. If both are the same, no need for trading at any point. If preferences where the same, but technology of intermediaries is superior, they would grow

away from their constraints. If technology where the same, but preference different, then there would be a transition towards wealth dominating for more patient guy. Interesting things occur when technology AND preferences are there.

3. Finally, and very importantly, constraints critically depend on prices. In the case where $\delta \to 1$, as He and Krishnamurthy, or Brunnermeier-Sannikov is the fact that constraints depend on prices. This is not true in other models, however. KM 97 true, KM 08 no. This leads to a pecuniary externality. Agents may fail to internalize that scale is to large. Planner may reduce volatility. Adrian and Boyarchenko's model has a second source of prices. That's the value of debt. On one hand, the price of debt falls quickly. On the other hand, the amount of assets that intermediaries receive, is lower. This generates a feedback loop.

Also, in AB, this induces a potential possibilities for defaults, which is interesting initself but I'm abstracting from this effect here. BS, also allow for equity positions by banks, but I'm abstracting from this aswell. At this stage, one can also realize the sources of inefficiencies in the model. These are two, (1) distortions in investment and capital missallocation.

3 Equilibrium

Definition of Equilibrium.

Definition 1 (Recursive Competitive Equilibrium)**.** *A recursive competitive equilibrium (RCE) is (1) are price functions, $q(X), p(X)$, (2) a set of policy functions for households and intermediaries: $\left\{c^j(b,k,X), k'^{,j}(b,k,X), b'^{,j}(b,k,X)\right\}_{j=}$ $i^i(b,k,X)$ (4) a law of motion for the aggregate state X: (I) The policy functions are solutions to the corresponding problems taking $q(X)$, $p(X)$ and the law of motion for X as given. (II) The market for capital clears (III) The bond market clears. (IV) The goods market clears and (V) The law of motion X is consistent with policy functions. Expectations are consistent with this law of motion.*

3.1 Solving for i_t

It is clear that the decision to solve for i only depends on prices. Assume that the constraint takes the typical form:

$$\max_i \left(p(X)\Phi(i/k) - i/k\right)k$$

so the investment to capital ratio is given by:

$$\Phi' = \frac{1}{p(X)}$$

So the investment to capital ratio solves:

$$i/k = (\Phi')^{-1}\left(\frac{1}{p(X)}\right).$$

A popular parametric accumption about Φ' are power forms which yields:

$$i/k = p(X)^{\upsilon}.$$

In such cases, investment solves:

$$
\begin{aligned}
I_t &= \left(p\left(X\right)^{\upsilon}\right)\int_0^1 k\left(i\right)di \\
&= p\left(X\right)^{\upsilon} K^i.
\end{aligned}
$$

This already captures the dynamic effects on constraints in the intermediaries. If intermediaries are forced to sell part of their capital stock, $q\left(X\right)$ will have to fall to clear out markets. However, this will put more downward pressure in prices and this will feedback. The concavity of ν will imply an ever growing reduction in I_t but eventually a desire to accumulate capital will sustain positive prices.

3.2 Demand for Assets and Price Determination

The linear structure of the problem provides a useful method to solve the model. Homothetic preferences and linear returns guarantee that policy functions are linear in wealth. There are several cases:

Log-linear. Housheolds are logarithmic, intermediaries are linear. In this case, intermediaries have no consumption up to a point where their expected wealth increase equals β.

Log-log. Both agenter are logarithmic. $(1-\beta)$ consumption shares. With more genereal CRRA preference specification, we can obtain a recursion in their savings rates.

Linear-linear. No insurance motives. Agents are hitting corners.

For the rest of these notes, I assume log-log. Now, we need to define some return functions. To make further progress, we need to define some additional objects:

$$
\begin{aligned}
R_k^i(X,X') &= \left[\frac{A^i\left(X'\right)-p\left(X'\right)^{\upsilon}+p(X')\left(\Phi\left(p\left(X'\right)^{\upsilon}\right)+\lambda\left(X'\right)\right)}{p(X)}\right], \\
R_k^h(X,X') &= \left[\frac{A^h\left(X'\right)+p(X')\lambda\left(X'\right)}{p(X)}\right] \\
R_b^i(X,X') &= \left[\frac{\delta+q(X')\left(1-\delta\right)}{q(X)}\right] \\
R_b^i(X,X') &= \left[\frac{\delta+q(X')\left(1-\delta\right)}{q(X)}\right]
\end{aligned}
$$

So the return to capital measured in consumption units has two components. The first is a physical risk, associated with the depreciation risk of capital units. The second term measures the volatility of prices. This is the discrete time version of the return equation in Brunnermeir-Sannikov.

The following proposition, summarizes the policy functions for all agents:

Proposition 1. *In any RCE, policy functions are* $c^l\left(k,b,X\right)=\left(1-\beta\right)n^j\left(k,b,X\right)$. *Morover, asset holdings are given by:*

$$
\begin{aligned}
p(X)k^{j,\prime}\left(k,b,X\right) &= \xi^j\left(X\right)\beta n^j\left(k,b,X\right). \\
q(X)b^{j,\prime}\left(k,b,X\right) &= \left(1-\xi^j\left(X\right)\right)\beta n^j\left(k,b,X\right)
\end{aligned}
$$

and

$$\underline{\textit{Household's}} \qquad\qquad \underline{\textit{Intermediaries's}}$$

$$\xi^{h}(X) = \arg\max_{\xi^{h}} \mathbb{E}\left[\log\left(\xi^{h} R_{k}^{h}(X,X') + \left(1-\xi^{h}\right) R_{b}^{h}(X,X')\right) | X\right] \quad \xi^{i}(X) = \arg\max_{\xi^{i}} \mathbb{E}\left[\log\left(\xi^{i} R_{k}^{i}(X,X') + \left(1-\xi^{i}\right) R_{b}^{i}(X,X'\right.\right.$$

$$\textit{subject to} \qquad\qquad \textit{subject to}$$

$$\xi^{h} \geq 0 \qquad\qquad \xi^{i} \geq 0$$

$$-\left(1-\xi^{i}\right) \leq \theta(X)\, q(X)$$

$$\hookrightarrow b^{i}(X) \leq \theta(X)\, n^{i}(X)$$

So we should be able to obtain a recursive representation for prices from the market clearing conditions. First, we use market clearing in debt markets. This yields:

$$\begin{aligned} \frac{\left(1-\xi^{h}\right) N^{h}}{q(X)} + \frac{\left(1-\xi^{i}\right) N^{i}}{q(X)} &= 0 \\ \hookrightarrow \quad 0 &= \left(1-\xi^{h}\right) + \left(1-\xi^{i}\right)\omega' \end{aligned} \tag{1}$$

and from the capital market clearing condition:

$$\frac{\xi^{h}}{p(X)} N^{h} + \frac{\xi^{i}}{p(X)} N^{i} = \left(p(X)^{\upsilon} + \lambda^{i}(X)\right) K^{i}(X) + \lambda^{h}(X) K^{h}(X) \tag{2}$$

and from consumption:

$$\left(1-\beta^{h}\right) N^{h} + \left(1-\beta^{i}\right) N^{i} = A^{i}(X) K^{i}(X) + A^{h}(X) K^{h}(X). \tag{3}$$

A useful computation is a recursive representation of equity for both groups. These are given by:

$$N^{h,\prime} = \left[\beta\left(R_{k}^{h}\xi^{h} + \left(1-\xi^{h}\right) R_{b}^{h}\right) - (1-\beta)\right] N^{h}$$

and

$$N^{i,\prime} = \left[\beta\left(R_{k}^{i}\xi^{i} + \left(1-\xi^{i}\right) R_{b}^{i}\right) - (1-\beta)\right] N^{i}.$$

Hence,

$$\omega' = \frac{N^{i,\prime}}{N^{h,\prime}} = \frac{\left[\beta\left(R_{k}^{i}(X,\phi')\,\xi^{i}(X) + \left(1-\xi^{i}(X)\right) R_{b}^{i}(X,\phi')\right) - (1-\beta)\right]}{\left[\beta\left(R_{k}^{h}(X,\phi')\,\xi^{h}(X) + \left(1-\xi^{h}(X)\right) R_{b}^{h}(X,\phi')\right) - (1-\beta)\right]}\omega \tag{4}$$

Thus, we need to solve for 4 unknowns, $\xi^{i}(X), \xi^{h}(X), p(X), q(X)$ at each possible state, taking future values as given.

3.3 Special Case

The papers we discussed are written in continuous time. Continuous time models with Brownian innovations have a particular structure that render their solution easier. In particular, they allow to drop the shocks from the state varaibles. With some redifinitions the model can be solved easily. First, one thing one can define capital in terms of efficiency units brought from the last period. Thus, $\tilde{k}_t = A_{t-1}k_t$.

To relate this structure to continuous time models, we can follow a binomial approximiation. We assume A_t follows a binary growth structure:

$$A_t^j = \begin{cases} (1+\Delta)\, A_{t-1}^j \text{ with probability } p \\ (1-\Delta)\, A_{t-1}^j \text{ with probability } (1-p) \end{cases}.$$

Later we impose take some limits that deliver A_t to become Geometric Brownian Motion process. With the change of vairiables, $p(X)$ must be adjusted accordingly via change of variables:

$$\tilde{p}(X) = p(X)/A(X).$$

In the portfolio problem above, we devide by A and obtain:

$$\begin{aligned} R_k^h(X, X') &= \left[\frac{A^h(X')/A + \tilde{p}(X')A^h(X')/A\lambda(X')}{\tilde{p}(X)}\right] \\ &= A^h(X')/A\left[\frac{1+\tilde{p}(X')\lambda(X')}{\tilde{p}(X)}\right] \end{aligned}$$

so the process yields:

$$= (1\pm\Delta)\left[\frac{1+\tilde{p}(X')\lambda(X')}{\tilde{p}(X)}\right].$$

Similarly for intermediaries we obtain:

$$(1\pm\Delta)\,\frac{1-\tilde{p}(X')^{\upsilon}+\tilde{p}(X')\left(\Phi\left(\tilde{p}(X')^{\upsilon}\right)+\lambda(X')\right)}{\tilde{p}(X)}$$

if the form of the adjustment costs is adapted accordingly.

3.3.1 Market Clearing Conditions

Goods. Let's re-examine the goods market clearing condition. It becomes:

$$\begin{aligned} \left(1-\beta^h\right)N^h + \left(1-\beta^i\right)N^i &= (1\pm\Delta)\left(\tilde{K}^i(X) + \tilde{K}^h(X)\right) \\ &= (1\pm\Delta)\left(\xi^i(X_{-1})\,N + \xi^j(X_{-1})\,N\right) \\ &= (1\pm\Delta)\left(\frac{\xi^i(X)\,N^h + \xi^j(X)\,N^i}{\tilde{p}(X)}\right) \end{aligned}$$

Now, current equity can be replaced succintly via the following formula:

$$\begin{aligned} &\left(1-\beta^h\right)\left(R_k^h(X,\Delta)\xi^h(X) + R_b^h(X,\Delta)\left(1-\xi^h(X)\right)\right)N^h \\ &+\left(1-\beta^i\right)\left(R_k^i(X,\Delta)\xi^i(X) + R_b^i(X,\Delta)\left(1-\xi^i(X)\right)\right)N^i \\ = \;& (1\pm\Delta)\left(\frac{\xi^i(X)\,N^h}{\tilde{p}(X)} + \frac{\xi^j(X)\,N^i}{\tilde{p}(X)}\right) \end{aligned}$$

Dividing both sides by N^h yields:

$$\begin{aligned}
&\left(1-\beta^h\right)\left(R_k^h(X,\Delta)\xi^h(X)+R_b^h(X,\Delta)\left(1-\xi^h(X)\right)\right)\\
&+\left(1-\beta^i\right)\left(R_k^i(X,\Delta)\xi^i(X)+R_b^i(X,\Delta)\left(1-\xi^i(X)\right)\right)\omega\\
=\;&(1\pm\Delta)\left(\frac{\xi^i(X)N^h+(1-\iota)\xi^j(X)\omega}{\tilde{p}(X)}\right) \qquad (5)
\end{aligned}$$

Where $\omega = N^i/N^H$. This shows that the market clearing condition is a function of the current state X, and ω. We don't know what that state is yet.

Bonds. The bond market clearing condition remains uncanged:

$$0=\left(1-\xi^h\right)+\left(1-\xi^i\right)\omega'. \qquad (6)$$

Capital. Finally, the capital market clearing condition is obtained following very similar steps:

$$\beta^h\frac{\left(R_k^h(X,\Delta)\xi^h(X)+R_b^h(X,\Delta)\left(1-\xi^h(X)\right)\right)}{\tilde{p}}+\frac{\xi^i}{\tilde{p}}\left(R_k^i(X,\Delta)\xi^i(X)+R_b^i(X,\Delta)\left(1-\xi^i(X)\right)\right)\omega \qquad (7)$$

$$=\left(\frac{\lambda^h(X)+\left(\tilde{p}^v+\lambda^i\right)\xi^i(X)\omega}{\tilde{p}(X)}\right) \qquad (8)$$

so arranging terms yields and expression for the current price of capital $\tilde{p}$.

This is an excellent example of the L&S quote that finding the state is an art. Assume that $X=\omega$. Then, observe that indeed, the entire system can be expressed in terms of ω. The value of Δ does not enter anywhere in the problem. Thus, we have to solve, for every value of X, prices $q(X'), q(X')$ such that $\xi^h(X), \xi^i(X)$ solve:

<u>Household's</u>

$$\xi^h(X)=\arg\max_{\xi^h}\mathbb{E}\left[\log\left(\xi^h R_k^h(X,\Delta')+\left(1-\xi^h\right)R_b^h(X,\Delta')\right)|X\right]$$

subject to

$$\xi^h\geq 0$$

<u>Intermediaries's</u>

$$\xi^i(X)=\arg\max_{\xi^i}\mathbb{E}\left[\log\left(\xi^i R_k^i(X',\Delta')+\left(1-\xi^i\right)R_b^i(X,\Delta\right.\right.$$

subject to

$$\xi^i\geq 0$$

$$-\left(1-\xi^i\right)\leq\theta(X)\,q(X,\Delta)$$

and (**??**),(**??**) and (**??**) which respect the law of motion of:

$$\omega'=\frac{N^{i,\prime}}{N^{h,\prime}}=\frac{\left[\beta\left(R_k^i(X',\Delta')\xi^i(X)+\left(1-\xi^i(X)\right)R_b^i(X',\Delta')\right)-(1-\beta)\right]}{\left[\beta\left(R_k^h(X',\Delta')\xi^h(X)+\left(1-\xi^h(X)\right)R_b^h(X',\Delta')\right)-(1-\beta)\right]}\omega.$$

We obtain from (**??**):

$$q(X)=\frac{1}{\theta(X)\omega}$$

and from the goods market clearing:

$$\left(1-\beta^h\right)N^h+\left(1-\beta^i\right)N^i=A^i(X)K^i(X).$$

From the evolution of equity:

$$\omega' = \frac{N^{i,\prime}}{N^{h,\prime}} = \frac{\left[\beta\left(R_k^i\left(X,\phi'\right)\xi^i(X) + \left(1-\xi^i(X)\right)R_b^i\left(X,\phi'\right)\right) - (1-\beta)\right]}{\left[\beta R_b^h\left(X,\phi'\right) - (1-\beta)\right]}\omega \tag{9}$$

Taking functions $q(\omega)$,$p(\omega)$ as given, this is a system of 4 variables: ξ^i, ξ^h, p, q and 4 equations.

Algorithm. Guess $q(\omega)$,$p(\omega)$. Then, then guess ξ^i, ξ^h. Solve p, q from market clearing conditions. Update $R_b^i, R_k^h, R_b^i, R_k^h$. Solve ξ^i, ξ^h. Then update until convergence inner loop. Then update until convergence outer loop.

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Portfolio Problems under Binomial Assumption:

$$p\log\left(\xi^j R_k^j(X,\Delta) + \left(1-\xi^j\right)R_b^j(X,\Delta)\right) + (1-p)\log\left(\xi^j R_k^j(X,-\Delta) + \left(1-\xi^j\right)R_b^j(X,-\Delta)\right)$$

Is there a convenient way to solve these FOCS?

4 Efficiency and Externalities

4.1 Constraints

Limited enforcement constraint, $\boldsymbol{\varphi}(X,X') > \bar{\boldsymbol{\varphi}}$. There's a double moral-hazard problem. Assuming intermediaries lend capital to entrepreneurs who manage these units. Entrepreneurs may diver funds $d_t \in \left[0,\bar{d}\right]$ for private benefit. Intermediaries can spend resources m_t monitoring the entrepreneur. By this, the private benefit for the entrepreneur is $\Xi(m_t)\,b_t$ where Ξ' is a decreasing convex function and $\Xi(m_t) < 1$ so that diversions aren't profitable.

Then, upon an allocation (b_t, m_t) of monitoring and benefits, capital evolves according to:

$$k_{t+1} = xxx$$

$\mathbf{n'} \geq \mathbf{0}$.

Limited Liability. By assumption, it is pressumed that intermediaries face a limited liability constraint by which, under every contract, $n + \varphi(X')\,\Pi > 0$. This constraint can be easily founded introducing a Moral-Hazard problem.

$$n - c + \varphi(X,X')\left[\Pi(X,X') - E\left[\Pi(X,X')\right]\right] + E\left[\Pi(X,X')\right] \geq 0$$

5 Binomial Approximation