Lecture 2: Limited Enforcement and Investment

These notes generalize the real side of the model of Nobuhiro Kiyotaki and John Moore (2012). That model studies the role of limited enforcement and exogenous shocks to the liquidity of assets that serve as collateral are that can be sold. The model lacks a second asset that competes with collateral, but we can build towards that goal later.

I the model is specialized to admit a balanced growth path as in a standard AK growth model (like in Acemoglu, Ch. 11) with a financial friction. Else, it specializes to a stationary environment.

1 Environment

Time is discrete and infinite horizon, $t = 1, 2, 3, ...$ A stochatic event is represented by s_t and the history of events is the vector of realizations of s_t from zero to t. We denote it by s^t.

Demographics and Preferences

Households. There are two types of households, entrepreneurs and workers —this distinction is made only to isolate wealth effects from labor supply decisions. I denote a household's type by the variable $h \in \{e, w\}$ where e and w denote an entrepreneur and worker respectively.

All households have the following utility function:

$$U\left(C_t^h, H_t^h\right) = \max_{\{C_t, H_t\}} \ln\left(C_t^h - \bar{v}_t^h \frac{\left(H_t^h\right)^{1+\nu}}{1+\nu}\right).$$

Here, $\bar{v}_t^h$ is a scalar that captures demographic variables that affect the labor supply of either sector. We let it vary over time when we want a balanced growth path.

Household's maximize expected discounted utility:

$$E\left[\sum_{t\geq 0} \beta^t U\left(C_t^h, H_t^h\right)\right].$$

and satisfy the following budget constraint:

$$C_t^h + q_t K_{t+1}^h = \left(1-\tau_k^h\right) \underbrace{r_t^H K_t^h + q_t \lambda K_t^h}_{\text{Capital Earnings}} + \left(1-\tau_w^h\right) \underbrace{w_t H_t^h}_{\text{Labor Earnings}}$$

where q_t is the price of capital, r_t the return per-unit-of capital, w_t the wage and λ the gross depreciation of capital. The tax rates $\left(\tau_k^h, \tau_w^h\right)$ are conveniente devices specialize the model.

Case 1. *When $\tau_k^h = 0, \tau_w^h = 0$, there are no distinction between households of different types. Thus, there's a representative household.*

Case 2. *When $\tau_k^e = 0$, $\tau_w^e = 1$, entrepreneur's don't supply labor. When $\tau_k^w = 1$, $\tau_w^w = 0$, workers are hand to mouth —they don't save.*

Note. In Kiyotaki and Moore (2012), the model is specialized to $(\tau_k^e = 0, \tau_w^e = 1, \tau_k^w = 1, \tau_w^w = 0)$.

Question. Jaimovic-Rebelo noted that models with GHH preferences do not admit a balanced growth path. The introduce an external habit that plays the role of $\bar{v}_t^h$. What is it?

Technology

Consumption–Good Producing Firms. A fringe of competitive firms produce consumption according to the following production function:

$$F(K_t, H_t) = K_t^\gamma H_t^\alpha.$$

The production function accomodates to two limit cases of interest.

Case 3. *When $\gamma = 1$, the production features constant-returns-to-scale in capital as in the standard AK model, except that there's still a role for a labor supply decision if α. This case admits a balanced growth path with the external habit.*

Case 4. *When $\gamma = 1 - \alpha$ the production features constant returns to scale in both capital and labor. This environment does not admit sustained growth and is consistent with a stationary equilibrium.*

Capital is rented and labor is supplied by household's. Firms buy these inputs in competitive markets. Thus, the cost structure for the firm is given by:

$$X_t(K_t, H_t) = r_t^C K_t + w_t H_t.$$

Note that X_t features a time subscript because the return to capital and labor are functions of time.

Firms maximize static profits:

$$\max_{\{K_t, H_t\}} F(K_t, H_t) - X_t(K_t, H_t).$$

Capital–Goods-Producing Firms. Capital-goods producing firms create capital with the use of —weakly— convex technologies. To create a new unit of capital, the capital-goods firm requires capital and consumption goods. Given an amount of capital k, the firm requires $\varphi(i,k)$ units of consumption goods to generate i units of capital. The convex cost $\varphi(i,k)$ is homogeneous and $\varphi(0,k) = 0$.

There are some cases of interest:

Case 5. *When $\varphi(i,k) = i$, the production of capital requires no capital.*

Case 6. *When $\varphi(i,k) = \varphi(i/k,1)\,k$, homogeneity of degree 1, the production cost depends on the investment rate.*

The first case corresponds to the case analyzed by Kiyotaki and Moore (2012). The role of convex adjustment costs is to introduce an additional source of price variation to the price of capital, q_t, which is independent of the financial frictions. This aspect is important for reasons we will discuss going forward.

Limited Enforcement. Note that when the firm produces capital, it requires consumption goods it does not posses. There is a hidden timing aspect that is not spelled out in the typical business cycle model without frictions.

Consumption goods are an intermediate input. To introduce limited enforcement in a tractable way, assume that capital goods firms use the capital of a particular household —they partner together in crime. However, the firm cannot use consumption goods owned by the same household. Instead, assume they either borrow the consumption goods needed to produce, or sell something to buy those goods from another household or a mix of both.

A loan contract is a promise to deliver i^s capital goods in exchange for consumption goods before production. This loan is subject to limited enforcement. In particular, the firm has *a technology* to divert funds. In particular it can run away with a fraction θ_t of the production of i capital goods. This possibility imposes some constraints on the contracts that can be signed.

There are two equivalent arrangements. In the one arrangement the firm can use a fraction ϕ_t of the capital stock as collateral. In the other arrangement, the firm sells the capital immediately against consumption goods. Let's show that both enviroments are isomorphic from the standpoint of the firm.

Financial Contracts - Capital as Collateral. When the firm pledges the fraction ϕ_t as collateral, it essentially sells claims in amount i^s against its future output. If the price of capital is q_t, the firm obtains $q_t i^s$ in consumption goods it can use to finance the production costs. If the firm runs away with the fraction θ_t of its output, then the fraction ϕ_t of the current capital is lost.

Problem 7. *The capital producing firm solves the following problem:*

$$R_t^K(k_t) = \max_{i,i^s} i - i^s$$

subject to:

$$q_t i^s = \varphi(i,k) \qquad \text{(Resource Constraint)}$$

and

$$i - i^s + \phi_t \lambda k_t \geq \theta_t i. \qquad \text{(Incentive Compatibility)}$$

The first constraint states that promises are sold at the market price. The second is the incentive compatibility constraint induced by limited liability. It states that the contract must be such that the firm is better-off staying in the contract than running away. The right-hand side is what the borrower gets to keep if he stays in the contract. That's the difference between total investment minus his promise, plus the fraction of collateral that can be pledged.

No lender would lend out to if will not receive any proceeds from the investment. If we get rid of i^s, the problem translates into:

Problem 8 (Equivalent Problem)**.**

$$R_t^K(k) = \max_i i - \frac{\varphi(i,k)}{q_t}$$

subject to:

$$i(1-\theta) + \phi_t k \geq \frac{\varphi(i,k)}{q_t}.$$

Before we solve this problem, we show what happens when the firm sells part of the old capital stock in advance instead of pledging capital as collateral.

Financial Contracts - Capital Sales. When the firm sells the fraction ϕ_t of its capital stock to obtain consumption goods for its investment projects, it relaxes the financial constraints imposed by limited enforcement. Assume thus that the firm sells that fraction at price q_t. Together with those sales, it also sells claims in amount i^s against its future output. The firm obtains $q_t(i^s + \phi_t \lambda k_t)$ in consumption goods it can use to finance the production costs. Again, If the firm runs away with the fraction θ_t of its output, then the fraction ϕ_t of the current capital is lost.

Problem 9. *Using capital sales, the capital producing firm solves the following problem:*

$$R_t^K(k_t) = \max_{i,i^s} i - i^s - \phi_t \lambda k_t$$

subject to:

$$q_t(i^s + \phi_t \lambda k_t) = \varphi(i,k)$$

and

$$i - i^s \geq \theta_t i.$$

Note that in this case, the objective has changed to incorporate the fact that when the firm sells capital, it loses the fraction sold from its capital stock. However, the enforcement constraint has changed because there is no capital left-over as collateral. Again, I can get rid of i^s by using the resource constraint of the firm. Substituting i^s out of the objective one obtains: $i - \varphi(i,k)/q_t$ as the objective. Replacing the same object into the incentive constraint, one obtains: $i(1-\theta) + \phi_t k \geq \frac{\varphi(i,k)}{q_t}$.

Exercise 10. *Show that when capital is sold, the total sales of capital (including both, existing capital and promises of capital) equals the amount of future capital in the version where capital is pledged as collateral. Show that when capital is sold, the firm retains the same amount of capital as when the firm pledges capital as collateral.*

Remark 11. *We have shown that selling capital or pledging capital as collateral is isomorphic. In particular, the firms solves the same problem of investment scale and the supply of capital by the firm is the same.*

Linear Returns to Capital Producing firms. Note that both the objective and the constraint faced by the capital producing firms are linear functions of k_t. Thus,

$$R_t^K(k) = r_t^K k$$

where

$$r_t^K = \max_{\iota} \iota - \frac{\varphi(\iota, 1)}{q_t}$$

subject to:

$$\iota(1-\theta) + \phi_t \geq \frac{\varphi(\iota, 1)}{q_t}.$$

Given an equilibrium q_t, if the constraint is not binding, then, $\frac{\varphi'(\iota,1)}{q_t} = 1$. Otherwise, if constraints is not binding: $\iota(1-\theta) + \phi_t = \frac{\varphi(\iota,1)}{q_t}$. We can summarize the result as:

Proposition 12. *Given an equilibrium q_t the investment rate of the firm is given by:*

$$\iota = \min\left\{(\varphi')^{-1}\left(\frac{1}{q_t}\right), \iota^{cons}\right\}$$

where ι^{cons} is the weakly positive solution of $\iota(1-\theta) + \phi_t = \frac{\varphi(\iota,1)}{q_t}$.

By assumption of convexity, there always exists a positive root in the solution ι^{cons}. It is worth to describe graphically what is going on.

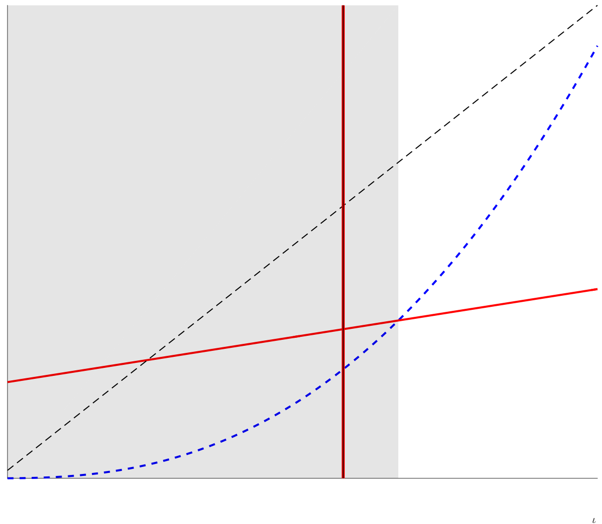

Figure 1: Unconstrained Solution for Capital Producing Firm

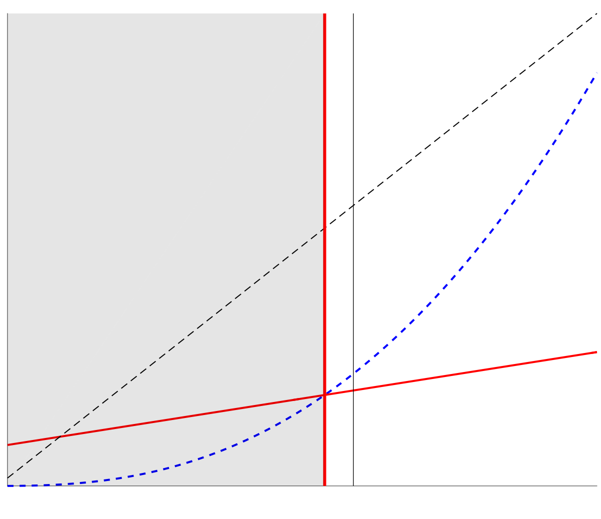

Figure 2: Constrained Solution for Capital Producing Firm

Return on Capital. Finally, we compute the return to household's capital. We define $r_t^H = \pi^K r_t^K q_t + \pi^C r_t^C$. When $\pi^K = 1 - \pi^C$, this expression has the interpretation that capital is devided into two sectors depending on the two fractions above. When $\pi^K = \pi^C = 1$, it is like assuming that capital is used twice. In Kiyotaki and Moore (2012), $\pi^K = 1 - \pi^C$. There is a distinction though, in that entrepreneur's are divided into two sectors with additional idiosyncratic risk. This distinction matters for asset pricing as it introduces a second form of risk.

Markets

1. There's a labor market in which firms hire hours.
2. There's a market for capital goods that clears are price $\{q_t\}$.
3. There's a market for collateralized investment loan contracts.
   - Loan contracts satisfy the incentive compatibility constraint.

2 Equilibrium

Definition. A sequential equilibrium is a sequence of prices $\{q_t, r_t^c, w_t\}$ and allocations $\left\{H_t^h, c_t^h, k_{t+1}^h, i_t^{s,h}, i_t^h\right\}_{t=0:\infty, h=\{e,w\}}$ su that the following conditions hold:

- Allocations are consistent with optimal decisions by both households,
- The labor, capital rental, and capital and investment markets clear at prices $\{w_t, r_t^c, q_t\}$.

3 Equilibrium - Characterization

In this section we solve the model for several special cases, we begin with the CRS on capital version —the simplest case.

Balanced Growth Path

Case 13. *Assume $\tau_k^h = 0, \tau_w^h = 0$ so there are no distinctions among households.*

Case 14. *Assume that the external habit $\bar{\upsilon}_t^h$ is proportional to the capital stock K_t and this is an internal habit.*

Case 15. *Assume $\gamma = 1$, so that the economy features constant returns to scale.*

The household j's FOC w.r.t. labor and consumption are:

$$\bar{v}_t^h H_t^{\nu} = w_t \left(1 - \tau^h\right).$$

Before solving the problem for consumption, we observe couple of things.

Observation 1. From the firm's problem, $w_t = \alpha K_t^{\gamma} H_t^{\alpha-1}$. Thus, $w_t H_t = \alpha K_t^{\gamma} H_t^{\alpha}$.

Observation 2. From observation 1, we have that: $K_t H_t^{\nu+1} = \alpha K_t^{\gamma} H_t^{\alpha} \rightarrow H_t^{(\nu+1-\alpha)} = \alpha$, thus

$$H^* = \alpha^{\frac{1}{\nu+1-\alpha}}.$$

Then, back in budget constraint, observe that including labor, the household's wealth is given by:

$$W_t = r_t^H K_t + w_t H_t + q_t \lambda K_t$$

We can open this constraint to get:

$$\begin{aligned} W_t &= r_t^H K_t + \alpha K_t^{\gamma} H_t^{\alpha} + q_t \lambda K_t \\ &= \left(q_t r_t^K + r_t^C + \alpha K_t^{\gamma-1} H_t^{\alpha} + q_t \lambda\right) K_t \\ &= \left(q_t \left(r_t^K + \lambda\right) + (1+\alpha)\left(H^*\right)^{\alpha}\right) K_t. \end{aligned} \tag{1}$$

This implies that household's maximize,

$$\max_{\{C_t, H_t\}} E \sum_{t \geq 0} \beta^t \left[\ln C_t - \frac{\alpha K_t H_t^{\alpha}}{1+\nu}\right].$$

subject to:

$$C_t + q_t K_{t+1} = \omega_t K_t.$$

Now, assume that the optimal labor choice has already been computed. Then, separability implies:

$$\max_{\{C_t\}} E \sum_{t \geq 0} \beta^t \ln\left(X_t\right)$$

where $X_t = C_t - \frac{H_t^{*1+\nu}}{1+\nu} = C_t - \frac{\alpha K_t H_t^{\alpha}}{1+\nu}$ where H_t^* is the optimal labor choice. Thus, we can write the budget constraint as:

$$X_t + \frac{\alpha K_t H_t^{\alpha}}{1+\nu} + q_t K_{t+1} = \omega_t K_t$$

and thus:

$$\begin{aligned} X_t + q_t K_{t+1} &= \left(q_t \left(r_t^K + \lambda \right) + (1+\alpha) H_t^{\alpha} - \frac{\alpha H_t^{\alpha}}{1+\nu} \right) K_t \\ &= \left(q_t \left(r_t^K + \lambda \right) + \frac{\nu(\alpha+1)}{1+\nu} (H^*)^{\alpha} \right) K_t \\ &= \omega_t K_t. \end{aligned}$$

The term ω_t is a wealth shifter that shifts the evolution of wealth as the price q_t and the returns of the investment firm move up and down.

Since this utility specification is the same as log utility plus random linear term, then, the policy functions satisfy the following conditions that we learned from lecture 2:

$$\begin{aligned} X_t &= (1-\beta)\,\omega_t K_t \rightarrow \\ C_t &= (1-\beta)\,\omega_t K_t + \frac{\alpha K_t H_t^{\alpha}}{1+\nu} \\ &= \left((1-\beta)\,\omega_t + \frac{\alpha^{\frac{\nu+1}{\nu+1-\alpha}}}{1+\nu} \right) K_t \end{aligned}$$

and

$$K_{t+1} = \frac{\beta \omega_t K_t}{q_t}.$$

Now, observe that in equilibrium:

$$K_{t+1} = (\iota_t + \lambda)\, K_t.$$

Substituting this expression in leads to the following condition:

$$(\iota_t + \lambda) = \frac{\beta \omega_t}{q_t}$$

which is independent of the capital stock.

Remark 16. *With constant-returns-to-capital, q_t depends on (ι_t, ω_t), but not on the capital stock.*

Wealth per-unit of capital is given $\omega_t = \left(q_t \left(r_t^K + \lambda \right) + (1+\alpha)\, \alpha^{\frac{\alpha}{\nu+1-\alpha}} \right)$. Now, observe that $r_t^K = \iota - \frac{\varphi(\iota,1)}{q_t}$ and thus:

$$\omega_t = \left(q_t \iota_t - \varphi\left(\iota_t, 1\right) + q_t \lambda + \frac{\nu\left(\alpha+1\right)}{1+\nu} \alpha^{\frac{\alpha}{\nu+1-\alpha}} \right)$$

Thus,

$$\begin{aligned} q_t\left(\iota_t+\lambda\right) &= \beta\omega_t \rightarrow \\ \left(1-\beta\right) q_t\left(\iota_t+\lambda\right) + \varphi\left(\iota_t, 1\right) &= \frac{\nu\left(\alpha+1\right)}{1+\nu} \alpha^{\frac{\alpha}{\nu+1-\alpha}} \end{aligned}$$

Observe that if the left-left hand side is increasing, we have a unique equilibrium investment and price of capital.

Question. Show that $\hat{\iota}$ is increasing in q_t . (Hint: employ the implicit function theorem.

Finally, in equilibrium,

$$\iota\left(q_t, \phi_t\right) = \min\left[\varphi'^{-1}\left(q_t\right), \hat{\iota}\left(q_t, \phi_t\right)\right] \tag{2}$$

where $\hat{\iota}\left(q, \phi\right)$ solves: $x\left(1-\theta\right) + \phi = \frac{\varphi(x)}{q}$.

An Equilibrium Subsystem. Thus, an equilibrium can be characterized by the following condition:

$$\left(1-\beta\right) q_t\left(\iota\left(q_t\right)+\lambda\right) + \varphi\left(\iota\left(q_t\right)\right) = \frac{\nu\left(\alpha+1\right)}{1+\nu} \alpha^{\frac{\alpha}{\nu+1-\alpha}}.$$

Once we solve for q_t from this equation, we can solve for ι_t as function of ϕ_t. Then, we can solve for C_t.

Analytic Example

Assume that φ is a quadratic of the form $\chi\frac{\iota^2}{2}$. Then, we obtain the following:

$$\varphi'\left(\iota\right) = \chi\iota \text{ and } \varphi'^{-1}\left(q\right) = q/\chi.$$

Also, $\hat{\iota}\left(q, \phi\right)$ is the solution to the following quadratic equation:

$$0 = \chi\frac{\iota^2}{2} - \left(1-\theta\right) q\iota - q\phi.$$

Employing the solution to a quadratic:

$$\hat{\iota}\left(q, \phi\right) = \frac{\left(1-\theta\right) q + \sqrt{\left(\left(1-\theta\right) q\right)^2 + 2\chi q\phi}}{\chi}$$

Thus, there is a minimal price q_t above which investment is unconstrained. It is the one that satisfies:

$$\frac{q}{\chi} = \frac{(1-\theta)\,q + \sqrt{((1-\theta)\,q)^2 + 2\chi q\phi}}{\chi}$$

Thus, we have:

$$q\theta = \sqrt{((1-\theta)\,q)^2 + 2\chi q\phi}.$$

Solving the quadratic:

$$\begin{aligned}(q\theta)^2 &= q^2 - 2\theta q + (q\theta)^2 + 2\chi q\phi \rightarrow \\ q^* &= 2\,(\theta - \chi\phi)\,.\end{aligned}$$

When does the constraint bind?

Suppose that invesetment is unconstrained. Then, we obtain: $\iota_t = q_t/\chi$ and the resources employed in investment are

$$\varphi\,(\iota\,(q_t)) = \frac{1}{2\chi}q_t^2$$

Thus, we check if q_t that satisfies the equilibrium price equation:

$$(1-\beta)\,q_t\left(\frac{q_t}{\chi} + \lambda\right) + \frac{1}{2\chi}q_t^2 = \frac{\nu\,(\alpha+1)}{1+\nu}\alpha^{\frac{\alpha}{\nu+1-\alpha}} \rightarrow 0$$

is indeed greater than q^*. Else, we employ the constrained solution:

$$q_t = \frac{\varphi\,(\iota_t)}{\iota_t\,(1-\theta) + \phi_t}.$$

In the market clearing condition:

$$(1-\beta)\,\frac{\varphi\,(\iota_t)}{\iota_t\,(1-\theta) + \phi_t}\,(\iota_t + \lambda) + \varphi\,(\iota_t) = \frac{\nu\,(\alpha+1)}{1+\nu}\alpha^{\frac{\alpha}{\nu+1-\alpha}}$$

This conditon again solves a quadratic.

Question. Solve this condition by obtaining a quadratic expression in q_t.

Liquidity Premium

We now modify the first model to allow for a risk free bond. Also, we extend the model to allow for Epstein-Zin preferences, but specialized to unit intertemporal elasticity of substitution. The household's problem is the following:

Household's maximize expected discounted utility:

$$u_t = \max_{\{C_t, B_t, K_{t+1}, H_t\}} \log\left(C_t^h - \bar{v}_t^h \frac{\left(H_t^h\right)^{1+\nu}}{1+\nu}\right) + \beta^t \log\left(\Psi^{-1}\mathbb{E}\Psi\left(\exp\left(u_t\right)\right)\right).$$

and satisfy the following budget constraint:

$$C_t^h + p_t B_{t+1} + q_t K_{t+1}^h = \left(1-\tau_k^h\right) r_t^H\left(B_t\right) K_t^h + q_t \lambda K_t^h + B_t + \left(1-\tau_w^h\right) w_t H_t^h.$$

We also assume that B_t is zero net supply.

Following the steps we learned in the previous lecture, we obtain the following:

$$\begin{aligned}
C_t &= (1-\beta)\,\omega_t K_t + \frac{\alpha K_t H_t^{\alpha}}{1+\nu} \\
K_{t+1} &= \beta \mu_t \omega_t K_t \\
B_{t+1} &= \beta\left(1-\mu_t\right)\omega_t K_t.
\end{aligned}$$

where μ_t is the portfolio weight.

The value of $r_t^H\left(B_t\right)$ is the sum of r_t^c as before but now the return to capital goods are::

$$R_t^K\left(k\,,b\right) = \max_{i,i^s} i - i^s$$

subject to:

$$q_t i^s = \varphi\left(i,k\right) \qquad \text{(Resource Constraint)}$$

and

$$i - i^s + b + \phi_t \lambda k_t \geq \theta_t i. \qquad \text{(Incentive Compatibility)}$$

The idea is that the new constraint now enables the capital producing firm to sell the bond to relax the constraint. Now we obtain $r_t^k\left(1\,,b/k\right)k$. This function is the one that solves:

$$r_t^K = \max_{\iota} \iota - \frac{\varphi\left(\iota,1\right)}{q_t}$$

subject to:

$$\iota\left(1-\theta\right)+\phi_t+b/k\geq\frac{\varphi\left(\iota,1\right)}{q_t}.$$

Since the asset is in zero net supply, we have the same formulas as before. However, we price the risk-less liquid bond according to:

$$\Omega_t=\max_{\mu_t}\Psi^{-1}\mathbb{E}\Psi\left[\frac{\left(R^h\left(\phi\right)\left(1-\mu_t\right)\mu_t+\left(1-\mu_t\right)R^b\right)}{\omega_t}\right].$$

Question. Derive the bond premium and decompose it into a risk premium and a liquidity premium.