Lecture 4: Inside and Outside Money

These lectures presents an economy in which there is an exogenous supply of money —outside money— which competes with privately issued IOUs —inside money— as a medium of exchange. Outside money can be thought of as commodity money —i.e., gold, silver— or a stock of government issued money.

Instead, IOUs are claims fully backed by physical capital which are pledged as collateral. IOUs and fiat money are essential as mediums of exchange because they relax financial frictions that follow from a limited enforcement problem that we learned in lecture 3. The novelty is that they affect the the purchase of production inputs, as opposed to capital. In this environment, fiat money has no intrinsic value. Nevertheless, money is valued in equilibrium because the quality of the capital that serves as collateral for issued IOUs is private information. Variations in the valuation of IOUs lead to variations in the value of inside money. Out goal is to study (a) the interaction of asymmetric information and limited enforcement and (b) how this introduces an endogenous trade-off between the issuance of IOUs which, ultimately, determines the equilibrium real-balances of both types of money.

The class has two goals. First, to provides a theoretical characterization on how fiat money interacts with IOUs in equilibrium in an economy with the features described. The second is to explain the effects of changes in the stock of money.

The key insights behind this notes follows from Kiyotaki and Moore [2002] theory of "Evil is the root of all fiat money.". This theory explains that fiat money is needed in this economy because promises to deliver a payment are not credibly pledgeable. Fiat money can be supplied either internally, for example when a commodity backs the issuance of IOUs. Their theory suggests that in presence of a constraint on the issuance of IOUs, an external object such as fiat fiat money in particular, will be valued as a medium of exchange. As we saw Kiyotaki and Moore [2008], the authors study how exogenous changes in the constraints that affect the issuance of IOUs have implications for the business cycle. In the economy studied here, such constraints arise endogenously do to a problem of asymmetric information. Here, we highlight that that the real value of fiat money and the issuance of IOUs are determined jointly, and that this has implications for monetary policy. By pumping more fiat money, a monetary authority distorts the incentives to issue IOUs.

The idea that inside and outside money are imperfect substitutes can be traced back to Gurley and Shaw (1960's) treaties on money but was not modeled explicitly. The model shares some features with classical cash-in-advance (CIA) economies such as **?**. We will see that here, fiat money here also serves the purpose of relaxing constraints. There is a key distinction. A related paper that belongs in the money search literature where that features competing mediums of exchange that differ in their information content is Rocheteau [2009].

1 Environment

1.1 Commodity and Asset Space

Time is discrete. The model is formulated in discrete time with an infinite horizon. There are two goods: a perishable consumption good —the *numeraire*— and capital. The aggregate capital stock is constant and equal to K. In addition, there are two assets, fiat money (outside money) and IOUs backed by physical capital. Every period there two aggregate shocks: first, there is a shock $\mu_t \in \mathbb{M}$ to the quantity of fiat money, and a shock $\xi_t \in \Xi \equiv \{\xi_1, \xi_2, ..., \xi_N\}$ to the distribution of capital quality. The shock ξ_t is an index over a family of distributions, $\{f_\xi\}$, that determines the evolution of capital units. The nature of this shock is explained later. The pair (μ_t, ξ_t) form a joint Markov process that evolves according to a transition probability $\Pi : (\mathbb{M}\times\Xi) \times (\mathbb{M}\times\Xi) \to [0,1]$ with the standard assumptions.

1.2 Demography

There is a continuum of entrepreneurs and set of workers. The measure of entrepreneurs and workers is each normalized to a unit.

Entrepreneurs. An entrepreneur is identified with a number $z \in [0,1]$. Entrepreneurs carry a capital stock, k_t, and fiat money holdings m_{t+1} across periods. Entrepreneurs use their capital for production and manage it without earning labor income from this activity. An entrepreneur's preferences over consumption streams are summarized by his expected utility:

$$\mathbb{E}\left[\sum_{t\geq 0} \beta^t U(c_t)\right]$$

where $U(c) \equiv \frac{c^{1-\gamma}}{1-\gamma}$, and c_t is the entrepreneur's consumption at date t.

Workers. Workers choose consumption and labor but don't save. Their period utility is given by:

$$U^w(c,l) \equiv \max_{c\geq 0, l\geq 0} c - L\frac{l^{1+\nu}}{(1+\nu)}$$

where l is the labor supply and c consumption. ν is the inverse of the Frisch-elasticity. Workers satisfy a static budget constraint in every period: $c = w_t l$ where w_t is their wage. L is just a normalization constant that is equivalent to changing the population of workers. The only role for workers in the model is to provide an elastic labor supply schedule.[1]

[1] This elasticity is not important for the results. If labor supply is inelastic, shocks have no effects on quantities but only on prices. One can also endow entrepreneurs with labor and omit workers. This specification, makes the analysis more complicated with further insights.

1.3 Technology

Production of consumption goods. An entrepreneur, z, carries out production using his capital stock, $k_t(z)$. By assumption, $k_t(z)$ is fixed at the time of production.[2] Capital is combined with labor inputs, $l_t(z)$, according to a Cobb-Douglas technology, $F(k,l) \equiv k^{\alpha}l^{(1-\alpha)}$, to produce output. The entrepreneur's profits are $y_t(z) = A_tF(k_t(z), l_t) - w_tl_t$ where w_t is the wage in period$t$.

Limited enforcement in labor contracts. There is limited enforcement in contractual relations between workers and entrepreneurs. Before carrying out production, z, hires an amount of labor promising to pay $w_tl_t(z)$ per unit of labor. The entire wage bill cannot be credibly pledged to be paid post-production because the entrepreneur may choose to default on his payroll. In that case, workers are capable to seize a fraction θ of output. In other words, z has a technology to divert the fraction $(1-\theta)$ of output for his own benefit without any further consequence.

The possibility of defaulting on labor contracts has two implications. First, it imposes a constraint on the entrepreneur's employment decision. This contrasts with an underlying assumption in RBC models where the wage bill is credibly paid after production. Second, it will induce entrepreneurs to sell part of their capital stock or obtain collateral with the purpose of relaxing the constraint.[3] This constraint is a milder version than the working capital constraints that require the entire wage bill to be paid up-front. Working capital constraints correspond to the case where θ is 0, and are used in Christiano et al. [2005] and Jermann and Quadrini [2009], to name some recent examples.[4]

Capital Accumulation. At the beginning of every period, capital held by each entrepreneur becomes divisible into a continuum of pieces. Each piece is identified by a quality $\omega \in [0,1]$. There is an increasing and differentiable function $\lambda(\omega) : [0,1] \to R_+$ that determines the corresponding future efficiency units remaining from a piece of quality ω by the end of the period. Efficiency units can also be interpreted as random depreciation shocks.[5]

The distribution of qualities assigned to each piece is randomly changing over time. In particular, at a given point in time, the distribution of capital qualities is determined by a density function f_{ξ}, which, in turn, depends on the current realization of the aggregate state ξ_t. To simplify the analysis, assume that the distribution is the same for every entrepreneur but differs across periods. Thus, the measure of units of quality ω is $k(\omega) = kf_{\xi}(\omega)$. Between periods, each piece is transformed into future efficiency units by scaling

[2] This assumption is really immaterial given that all entrpreneurs have the same productivity and the technology features constant returns to scale to capital. If allowed, entrepreneur's would not reallocate capital.

[3] Implicitly, it is assumed that workers are not willing to take capital as collateral directly because they are small. Financial firms are bigger and, therefore, are capable of fully diversifying risk.

[4] These constraints are found in earlier work by CEE1992, 1995 and Fuerst 1992. Mention Tobin's comment from CEE1992?

[5] Shocks to the efficiency units are commonly used in continuous time settings as **?**, for example. Other examples include the disaster literature or Gertler Kiyotaki for example.

the qualities by $\lambda(\omega)$. Thus, $\lambda(\omega)k(\omega)$ efficiency units remain from the $\omega-$qualities. Once capital units are scaled by depreciation, they are merged back into a homogeneous capital stock. Thus, by the end of the period, the capital stock that remains from k is,

$$\tilde{k} = \int \lambda(\omega)k(\omega)d\omega = k \int \lambda(\omega)f_{\xi}(\omega)d\omega.$$

In the following period, capital is divided again in the same way and the process is repeated indefinitely.

This does not mean that the entrepreneur will necessarily hold all of $\tilde{k}$ in subsequent periods. On the contrary, depending on their decisions, entrepreneurs will sell certain amounts of capital qualities. Let, $\phi(\omega; z, s^t) : [0, 1] \to [0, 1]$ represent the fraction of capital of quality ω that are sold by z given a history s^t. Thus, an entrepreneur will get rid of $k \int \lambda(\omega)(1 - \phi(\omega; z, s^t))f_{\xi}(\omega)d\omega$ units of capital . Accounting shows that the efficiency units that remain with the entrepreneur are $k \int \lambda(\omega)\phi(\omega; z, s^t)f_{\xi}(\omega)d\omega$. Including his purchases of $t + 1$ capital, z's capital stock evolves according to:

$$k_{t+1}(z) = x_t(z) + k_t(z) \int \lambda(\omega) \left[1 - \mathbb{I}(\omega)\right] f_{\xi_t}(\omega)d\omega, \tag{1}$$

$x_t(z)$, are the purchases of t+1 capital by the entrepreneur in period t.

I impose some structure on the quality distributions $\{f_{\xi}\}$:

The set $\{f_{\xi}\}$ satisfies the following:

1. For any $\xi \in \Xi$, $\int \lambda(\omega)f_{\xi}(\omega)d\omega = 1$.

2. $\mathbb{E}_{\xi}\left[\lambda|\lambda < \lambda^*\right] \equiv \int_{\omega<\lambda^{-1}(\lambda^*)} \lambda(\omega)f_{\xi}(\omega)d\omega$ is weakly decreasing in ξ for any λ^*.

The first condition states that for any ξ, the mean of the distribution of capital quality is always 1. That is, ξ-shocks are mean preserving shocks (MPS). The role of this condition is to make the process consistent with the fact that the amount of capital in this economy is constant. This can be seen by noticing that what remains of z's initial capital $k_t(z, s^t) \int \lambda(\omega)f_{\xi}(\omega)d\omega = k(z, s^t)$. Integrating across the entrepreneurs capital stocks, one can verify that the aggregate amount of capital is constant K.

The second assumption provides a meaning to the magnitude of these shocks. This second condition states that the expected quality under some threshold amount of capital is always decreasing in ξ. This condition has the implication of exacerbating asymmetric information, which for example, guarantees that pooling outcomes are worse when ξ is larger.

1.4 Information Structure

Information. The aggregate state of the economy is summarized by $s_t \equiv \{\mu_t, \xi_t\} \in \mathbb{X} \equiv \mathbb{M} \times \Xi$. s_t is known at the beginning of the period. In contrast, the $\omega-$quality behind a piece of capital is only known to the entrepreneur that owns it. This means that the amount of capital sold by an entrepreneur, $k \int \phi(\omega; z, s^t) f_\xi(\omega) d\omega$, is observable, but it is impossible to infer the value of $k \int \lambda(\omega)\phi(\omega; z, s^t) f_\xi(\omega) d\omega$.

Timing. The sequence of actions taken by the agents in this economy is as follows. At the beginning of each period all the relevant information is revealed. Then, entrepreneurs choose which qualities to sell. In exchange they obtain IOUs (payed in units of consumption) denoted by $n_t(z, s^t)$. These IOUs are backed by capital units. An implicit clearing house is in charge of settling these claims. The entrepreneur's total real balances are equal to $p^{-1}(s^t) m_{t+1}(z, s^t) + n_t(z, s^t)$ where $p^{-1}(s^t)$ is the price index.

Entrepreneurs use IOUs and there holdings of outside money to pay an upfront payment of a fraction $1 - \sigma$ of the salaries. Workers then provide labor and production takes place. Consumption goods are used by entrepreneurs to, (i) pay for the remaining fraction of the wage bill $(1 - \sigma) w_t$, (ii) to consume, and (iii) to repurchase capital. Finally, the claims to IOUs are settled by an invisible hand. For the rest of the paper, I treat these actions as if they occur simultaneously without further reference to the timing of events.

1.5 Monetary Policy Instruments

Note that there is no fiscal authority in this economy, so all the government can do is finance it's operations via seignioriage.

Interests on Monetary Reserves. Printing money is captured via the money growth rate μ_t which via the Markov matrix Π determines a monetary rule. Money can be injected into the economy in two different ways. The first ways is by paying providing a nominal interest on money holdings. The assumption here is that and entrepreneur that holds $\tilde{m}_{t+1}$ by the end of period t, will be payed $\left(\mu_{t+1}\right)$ in interests, wich the central bank finances by printing more money. Thus, by the beginning of the following period, an agent will have $m_{t+1} = \left(1 + \mu_{t+1}\right) \tilde{m}_{t+1}$ monetary reserves by the end of the period.

A rule that offsets a decline in liquidity is matrix negatively correlated with Π. A zero lower bound, for example, is a constraint that restricticts $\mu_{t+1} \geq 0$. A central bank can also choose to target a given interest rate. For example, it can set μ_{t+1}, to affect the price of a riskless bond in zero supply. I leave this possibility to a later section.

Open Market Operations. Open market operations are done by offering a given fixed supply of money in exchange for IOUs. Thus, the central bank adds an additional term to the demand for IOUs in the economy. If by by the end of the period, the units are resold, this operation corresponds to a REPO contract.

1.6 Allocations and Market Structure

Allocations. Denote by $s^t \equiv (s_1, s_2, ..., s_t) = \{\mu_t, \xi_t\}_{\tau=0}^{t}$, the history of aggregate shocks up to time t. An allocation is a set of functions, $c_w(s^t)$, $l(s^t)$, $c(z,s^t)$, $x(z,s^t)$, $k(z,s^t)$, $l(z,s^t)$, $m_{t+1}(z,s^t)$ and $\phi(\omega,z,s^t)$, that map histories and agents into quantities of consumption, labor, capital accumulation, labor input choice, fiat money holdings and binary sales decisions for the $\omega-$qualities. Allocations are *feasible* whenever the following conditions hold:

$$
\begin{aligned}
c_w(s^t) + \int_0^1 c(z,s^t)dz &= \int_0^1 F(k(z,s^{t-1}), l(z,s^t))dz \\
\int_0^1 x(z,s^t)dz &= \int_0^1 k(z,s^{t-1}) \int_0^1 \phi(\omega; z, s^t)\lambda(\omega) f_\xi(\omega) d\omega dz \\
\int_0^1 l(z,s^t)dz &= l(s^t) \\
\int_0^1 m_{t+1}(z,s^t)dz &= \mu(s_t) \int_0^1 m_t(z,s^t)dz
\end{aligned}
$$

for all possible s^t. An allocation is *incentive feasible* if it is feasible and satisfies,

$$
\begin{aligned}
(1-\sigma(z,s^t))w_t(s^t)l(z,s^t) &\leq p^{-1}(s^t)m_t(z,s^t) + n_t(z,s^t) \\
F(k(z,s^{t-1}), l(z,s^t)) - \sigma(z,s^t)w_t(s^t)l(z,s^t) &\geq (1-\theta)F(k(z,s^{t-1}), l(z,s^t)), \forall z, s^t
\end{aligned}
$$

Here, $\sigma(z,s^t)$, represents the fraction of the wage bill paid in advance by the entrepreneur.

Markets. Labor markets are perfectly competitive and clear at wage rate $w_t(s^t)$. The market for capital is sold in exchange for IOUs is descentralized. Following, Guerrieri, Shimer and Wright (2010) there is a continuum of markets for capital. Each market is characterized by a price $\tilde{q} \in \mathbb{R}_+$ and an endogenous probability $\phi(\tilde{q}, s^t)$ of the realization of a sale. Thus, $\phi(\tilde{q}, s^t)$ represents the probability that a unit of capital is sold in a $\tilde{q}-$market. This probabilities are equilibrium objects.

The probability $\phi(\tilde{q}, s^t)$ is pinned in the following way. When purchasing capital in a $\tilde{q}$ market, agents form expectations over the expected quality in a given market. After forming this expectation, agents decide on how many units to buy from each market. This decisions will pin down an amount of consumption goods brought into each market. Dividing this amount, by the price in each market, pins down, the amount of units that will be effectively transferred in each market.

In contrast, when selling, agents will choose on an amount of capital of each quality to bring into a $\tilde{q}$ market. Let $\gamma(\tilde{q}; \omega, z, s^t)$ represent the distribution of capital of quality ω brought to a $\tilde{q}$ market by z given a history s^t.

The ratio between the amount of consumption goods over $\tilde{q}$ times the amount of capital brought into each market will pin down $\phi(\tilde{q}, s^t)$. An agent wishing to accumulate x units of $t+1$ capital must pay a price q will solve the following problem:

$$\begin{aligned} q(s^t) &= \min_{\eta(\tilde{q})} \int_0^\infty \tilde{q}^{-1}\eta(\tilde{q})d\tilde{q} \text{ subject to} \\ 1 &= \int_0^\infty \mathbb{E}\left[\lambda|\tilde{q}, s^t\right]\eta(\tilde{q})d\tilde{q} \end{aligned}$$

where $\mathbb{E}\left[\lambda|\tilde{q}, s_t\right]$ represents the expected value of λ of the units sold in market $\tilde{q}$ given a history s^t. Thus, $q(s_t)$ is the effective cost of $t+1$-capital and is also the full information price of capital. Rational expectations requires that:

$$\mathbb{E}\left[\lambda|\tilde{q}, s^t\right] = \frac{\int_0^1 \int_0^1 \lambda(\omega)k(z, s^t)\gamma(\tilde{q}; \omega, z, s^t)d\omega dz}{\int_0^1 \int_0^1 k(z, s^t)\gamma(\tilde{q}; \omega, z, s^t)d\omega dz}$$

Thus, in equilibrium,

$$\phi(\tilde{q}, s^t) = \frac{\int_0^1 \tilde{q}^{-1}\eta(\tilde{q})x(z, s^t)dz}{\int_0^1 \int_0^1 k(z, s^t)\gamma(\tilde{q}; \omega, z, s^t)d\omega dz}$$

I assume and later verify for the applications of interest that $\phi(\tilde{q}, s^t) \in [0,1]$, so for now I don't worry about rationing over buyers. It is clear that $\phi(\omega, z, s^t) = \phi(\tilde{q}, s^t)\gamma(\tilde{q}; \omega, z, s^t)$.

Separating markets. A perfectly separating equilibria is one where $\omega-$quality is associated with a $\tilde{q}$ market. I denote this price by $\tilde{q}(\omega, s^t)$ the price associated with the ω quality and $\phi(\omega, s^t)$ the probability of selling a ω unit in that market.

Pooling markets. Under this market arrangement, there also exists pooling equilibria such that only a single $\tilde{q}$ market is open. In such cases I use $\tilde{q}(s^t)$ to denote this price. The characterization of these markets is presented in the subsequent section.

1.7 Agent Problems

The entrepreneurs problem is,

$$\max_{\gamma(\tilde{q},\omega,,z,s^t)c_t(z,s^t),x_t(s^t),m_{t+1}(s^t),l(z,s^t),\eta(\tilde{q},z,s^t),\sigma(z,s^t)} \mathbb{E}_0\left[\sum_{t\geq 0}\beta^t U(c(z,s^t))\right]$$

subject to

$$
\begin{aligned}
& c_t(z,s^t) + q(s^t)x_t(z,s^t) + p^{-1}(s^t)m_{t+1}(z,s^t) = \\
& \qquad\qquad F(k(z,s^{t-1}),l(z,s^t)) - w_t(s^t)l(z,s^t) + p^{-1}(s^t)m_t(z,s^t) + n_t(z,s^t) \\
& k(z,s^t) = x_t(z,s^t) + k(z,s^{t-1})\int_0^1 \lambda(\omega)[\int_0^\infty \gamma(\tilde{q},\omega,,z,s^t)(1-\phi(\tilde{q},s^t))d\tilde{q}]f_{\xi_t}(\omega)d\omega \\
& n_t(z,s^t) = k(z,s^{t-1})\int_0^1 [\int_0^\infty \tilde{q}\gamma(\tilde{q},\omega,z,s^t)(1-\phi(\tilde{q},s^t))d\tilde{q}]f_{\xi_t}(\omega)d\omega \\
& (1-\sigma(z,s^t))w_t(s^t)l(z,s^t) \leq p^{-1}(s^t)m_t(z,s^t) + n_t(z,s^t) \\
& F(k(z,s^{t-1}),l(z,s^t)) - \sigma(z,s^t)w_t(s^t)l(z,s^t) \geq (1-\theta)F(k(z,s^{t-1}),l(z,s^t))
\end{aligned}
$$

In the problem $q(s^t)$ and summarizes the optimal decision of optimal purchases in different markets — either for separating an pooling market arrangements. The first constraint is the entrepreneur's a budget constraint. The right hand side of the budget constraint corresponds to the entrepreneur's profits a plus his real balances. The second constraint, is the evolution of his capital stock which was explained earlier. The next equation, , says that the IOUs in hand are obtained by selling capital units in the different $\tilde{q}$ markets $n_t(z,s^t)$. The next equation says that the fraction of the entrepreneur's wage bill, $(1-\sigma(z,s^t))$ payed up-front is financed by using the entrepreneur's real balances. By the end of the period, the entrepreneur owes the $\sigma(z,s^t)$ fraction of the wage bill to the workers. Workers then return to the firm, and exchange the amount of real balances given in advance in exchange for $(1-\sigma(z,s^t))w(s^t)l(z,s^t)$ of the wage bill. For this reason, the budget constraint substracts the total wage bill from the entrepreneur's output but adds back the real balances.

The incentive compatibility constraint, (**??**), states that the amount of income after the entrepreneur pays the remaining part of the wage bill must exceed the amount of funds that can be diverted. Rational workers require this incentive compatibility because they can provide work to other entrepreneurs at the same wage.

Definition 1. *A competitive equilibria for this economy is a sequence of feasible allocations, prices* $p_t(s^t)$ *and* $q_t(s^t)$ *and wages* $w_t(s^t)$ *such that the following hold. (1) Allocations are consistent with the agents problem. (2) Labor markets clear. (3) Each capital markets clearing at* $\phi(z)$ *and* $q_t(s^t)$ *is a solution to with the agents problems. (4) The, goods markets clear. Finally, decisions satisfy rational expectations.*

2 Characterization

2.1 Optimal Employment Decisions

To characterize equilibria, it is convenient to break-up the problem into 3 parts. The strategy is the following. First, I solve for the optimal labor-capital ratio of the firm, given an amount of real balances. The solution to this problem shows up as the value of real balances and allows me to obtain an expression for the marginal value of real balances.

Then, I use the marginal value of real balances to find out what is the optimal amount of having liquid assets. This condition pins down $n_t(z, s^t)$. Finally, one can collapse the entire problem into a standard consumption-savings with a stochastic return on savings. This stochastic return is an endogenous object. This problem involves solving a portfolio problem that depends on the rate of inflation and the return to capital. Taking the law of motion for prices as given, this problem allows me to pin down, the capital accumulation equation and the total demand for fiat money. Using this demand functions one can pin down prices, using prices one pins down liquidity. Finally, one must look for a fixed point.

Let's begin by consider the problem of an entrepreneur with real balances r and 1 unit of capital facing a wage of w:

Problem 2. *The profit maximization problem is*

$$\pi(r, w) = \max_{l,\sigma} \left[l^{1-\alpha} - wl \right]$$

subject to $l^{1-\alpha} - \sigma wl \geq (1-\theta)l^{1-\alpha}$ *and* $(1-\sigma)wl \leq r$.

The solution to this problem is given by the following Lemma:

Lemma 3 (Optimal Labor)**.** *The solution to the profit maximization problem 2 is* $l^*(r, w) = \min \{l^{cons}(r, w), l^{unc}(w)\}$ *where* $l^{cons}(r, w) = \max \{l : \theta A l^{1-\alpha} + r = wl\}$ *and* $l^{unc}(w)$ *is the unconstrained labor choice given a wage* w*. Constraints are always slack if* $\theta \geq (1-\alpha)$*.*

Lemma 3 states that an entrepreneur may be constrained to hire less than the efficient amount of labor if he lacks sufficient liquidity. When liquidity is insufficient, both constraints will bind because there is no point in leaving liquid funds without use. If constraints bind, the entrepreneur is bound to choose employment so that his wage bill equals his liquid funds plus the pledgeable fraction of income. The max simply takes care of not choosing $l = 0$ when $x = 0$ because the entrepreneur can still hire workers even if $x = 0$. If, $\theta^L < (1-\alpha)$, efficient employment requires some amount of liquidity because the labor share of output is cannot be credibly pledgeable to workers.

Returning to the entrepreneur's problem, one can use the principle of optimality to get rid of $l(z, s^t)$ in the problem. Suppose, $m^*_{t+1}(s^t), \gamma^*(\tilde{q}, \omega, , z, s^t)$ and $\eta(\tilde{q}, z, s^t)$ are part of the entrepreneur's optimal plan. Then, the optimal choice of $\sigma(z, s^t)$ and $l(z, s^t)$ is summarized simply by the amount of liquidity and the capital stock of the entrepreneur. The following Lemma shows states this result formally.

Lemma 4 (Producer's Problem II)**.** *The problem of the entrepreneur is equivalent to:*

$$\max_{\gamma(\tilde{q},\omega,z,s^t) c_t(z,s^t), x_t(s^t), m_{t+1}(s^t), l(z,s^t), \eta(\tilde{q},z,s^t)} \mathbb{E}_0 \left[\sum_{t \geq 0} \beta^t U(c(z, s^t)) \right]$$

subject to:

$$c_t(z, s^t) + q(s^t) x_t(z, s^t) + p^{-1}(s^t) m_{t+1}(z, s^t) = [\pi(r_t(z, s^t), w(s^t)) + r_t(z, s^t)] k(z, s^t)$$

$$r_t(z, s^t) = \frac{p^{-1}(s^t) m_t(z, s^t) + n_t(z, s^t)}{k_t(z, s^t)}$$

$$k(z, s^t) = x_t(z, s^t) + k(z, s^{t-1}) \int_0^1 \lambda(\omega) \left[\int_0^\infty \gamma(\tilde{q}, \omega, , z, s^t)(1 - \phi(\tilde{q}, s^t)) d\tilde{q} \right] f_{\xi_t}(\omega) d\omega$$

$$n_t(z, s^t) = k(z, s^{t-1}) \int_0^1 \left[\int_0^\infty \tilde{q} \gamma(\tilde{q}, \omega, z, s^t)(1 - \phi(\tilde{q}, s^t)) d\tilde{q} \right] f_{\xi_t}(\omega) d\omega$$

where $\pi(r, w)$ is the value of Problem 2.

This lemma exploits the fact that employment does not directly affect any intertemporal decision. $r(x, X)k$ are the profits the entrepreneur obtains when he chooses labor optimally subject to constraints (**??**) and (**??**) for a given value of x.

It is convenient to save some notation an denote by $\tilde{\pi}(z, s^t) \equiv \pi(r_t(z, s^t), w(s^t))$. It is easy to verify that π has a well defined derivative which I denote by $\pi_r(r, w)$. This derivative is the marginal per-unit-of-capital profit obtained by incrementing liquid assets in 1 unit. The, rest of the characterization crucially depends on $\pi_r(r, w)$. I proceed explaining first equilibria under separating markets for the IOUs and then explain the pooling counterpart. Intermediate cases with bunching regions are not analyzed in this paper.

2.2 Endogenous Liquidity

Here we solve the model for both pooling and perfectly separating equilibria. We can show that propositions are independent of the market protocol.

2.2.1 Separating Equilibria

Separating equilibria. Assume an equilibrium is separating. Then we have the following properties:

Theorem 5 (Separating Equilibria)**.** *A separating equilibria must satisfy the following:*

$$\tilde{q}(\omega, s^t) = q(s^t)\lambda(\omega)$$

and

$$\phi(\omega, s^t) = \left[\exp\left(-\frac{(1+\pi_r(z,s^t))\,(\lambda(\omega)-\lambda(0))}{q(s^t)(1+\pi_r(z,s^t))-1}\right)\right]$$

and

$$\begin{aligned}\gamma(x,\omega,z,s^t) &= 1 \text{ } \textit{if } \tilde{q}(\omega,s^t) = x \\ \gamma(x,\omega,z,s^t) &= 0 \text{ } \textit{otherwise}\end{aligned}$$

Proof. This is a sketch. To pin down the price, argue that if it is separating, then $\lambda(\omega)$ is known. But then, agents must be indifferent. Then $q(s^t)$ is a simple market clearing condition price for the quantity-quality supply, and given a demand. To pin down the probability, write down the corresponding IC, and use the monotonicity, to argue that the solution is differentiable a.e. Then, use the property of local differentiation and the envelope theorem. Integrating over λ, one pins down the condition in the proposition. □

With this proposition, we can easily characterize the amount of inside liquidity held by the entrepreneur:

$$n_t\left(z,s^t\right) = \upsilon_t^{sep}\left(s^t\right)k_t\left(z,s^t\right)$$

where $\upsilon_t^{sep}\left(s^t\right) = q(s^t)\int_0^1 \phi(\omega,s^t)\lambda(\omega)f_\phi\left(\omega\right)d\omega \leq 1$

Back in the agents problem, we can use this proposition to assert that the entrepreneur's budget constraint is:

$$\begin{aligned}&c\left(z,s^t\right)+q(s^t)k_{t+1}\left(z,s^t\right)+p_t^{-1}\left(s^t\right)m_{t+1}\left(z,s^t\right)\\ =& \left(\tilde{\pi}\left(z,s^t\right)+\upsilon_t^{pool}\left(s^t\right)\right)k_t\left(z,s^t\right)+q(s^t)\int_0^1\left(1-\phi(\omega,s^t)\right)\lambda(\omega)f_\phi\left(\omega\right)d\omega+p_t^{-1}\left(s^t\right)m_t\left(z,s^{t-1}\right)\\ =& \left(\tilde{\pi}\left(z,s^t\right)+q(s^t)\right)k_t\left(z,s^t\right)+p_t^{-1}\left(s^t\right)m_t\left(z,s^{t-1}\right)\equiv W^{sep}\left(z,s^t\right)\end{aligned}$$

We return to this expression after we present an analogue result for pooling market arrangements.

2.2.2 Pooling Equilibria

Pooling equilibria. Let $\pi_r(s^t)k\left(z,s^t\right)$ be the marginal profit obtained by and extra unit of liquidity per unit of capital.

Theorem 6 (Pooling Equilibria). *A pooling equilibria must satisfy the following. There is a unique price* $\tilde{q}(\omega, s^t)$ *and*

$$\tilde{q}(\omega, s^t) = q(s^t)\mathbb{E}\left[\lambda(\omega)|\omega < \omega^*\right]$$

where

$$\omega^* \text{ solves } \lambda(\omega^*) = (1 + \pi_r(s^t))\mathbb{E}\left[\lambda(\omega)|\omega < \omega^*\right]$$

and

$$\phi(\omega, s^t) = 1 \text{ if } \omega < \omega^* \text{ and 0 otherwise}$$

With this proposition, we can easily characterize the amount of inside liquidity held by the entrepreneur:

$$n_t\left(z, s^t\right) = \upsilon_t^{pool}\left(s^t\right) k_t\left(z, s^t\right)$$

where

$$\upsilon_t^{pool}\left(s^t\right) = q(s^t)\int_0^{\omega^*} \lambda(\omega) f_\xi\left(\omega\right) d\omega \leq 1$$

Following similar steps as before, we obtain that the agents budgetc constraint (RHS) is:

$$\left(\tilde{\pi}\left(z, s^t\right) + q(s^t)\right) k_t\left(z, s^t\right) + p_t^{-1}\left(s^t\right) m_t\left(z, s^{t-1}\right) \equiv W^{pool}\left(z, s^t\right)$$

2.3 Policy Functions

This conditions is useful to pin down the policy functions for consumption, investment and real fiat money balances:

Proposition 7. *The policy functions of the entrepreneur are characterized as follows:*

$$\begin{aligned} c(z, s^t) &= (1-\beta) W\left(z, s^t\right) \\ k_{t+1}(z, s^t) &= \frac{\rho\left(z, s^t\right)(1-\beta)}{q(s^t)} W\left(z, s^t\right) \\ m_{t+1}(z, s^t) &= p_t\left(s^t\right)\left(1 - \rho\left(z, s^t\right)\right)(1-\beta) W\left(z, s^t\right) \end{aligned}$$

and ρ *solves:*

$$\rho\left(z, s^t\right) = \max_{\rho} \mathbb{E}\left[\log\left(\rho R^k\left(\rho, s^{t+1}\right) + (1-\rho) R^m\left(s^{t+1}\right)\right)|s^t\right]$$

where

$$
\begin{aligned}
R^{m}\left(s^{t+1}\right) &= \frac{p\left(s^{t}\right)}{p\left(s^{t+1}\right)}\mu\left(s^{t+1}\right) \\
R^{k}\left(\rho,s^{t+1}\right) &= \frac{q\left(s^{t+1}\right)+\pi\left(r\left(\rho,s^{t+1}\right),w\left(s^{t}\right)\right)}{q\left(s^{t}\right)} \\
r\left(\rho,s^{t+1}\right) &= q\left(s^{t}\right)\left[\frac{p\left(s^{t}\right)}{p\left(s^{t+1}\right)}\frac{(1-\rho)}{\rho}+\frac{q\left(s^{t+1}\right)}{q\left(s^{t}\right)}\nu\left(s^{t+1}\right)\right]
\end{aligned}
$$

The proposition shows that policy functions are linear in the entrepreneur's wealth. The proposition also shows that the portfolio weight $\rho\left(z,s^{t}\right)$ on capital solves a portfolio problem in two returns. $R^{m}\left(s^{t+1}\right)$ is the return on fiat money. This return equals the inverse of inflation plus the shadow value of fiat money. The shadow value of fiat money is the marginal profit per-unit of capital obtained by having an extra unit of cash. In contrast, $R^{k}\left(s^{t+1}\right)$, the return on capital, equals its common expression: future returns and dividends plus the shadow value of capital. Since $\upsilon_{t}\left(s^{t}\right)\leq 1$, we now that the liquidity value of capital is less than the liquidity value of fiat money. The shadow value of capital is adjusted by the fact that capital is less liquid than fiat money due to the informational asymmetries.

Appendix xxx provides a sketch of the numerical algorithm used to solve the model.

2.4 Positive Results

A couple of results are worth mentioning.

Frictions. In this economy, there's a tension between relaxing financial constraints and the solution to the problem of selling capital under asymmetric information. Consider the case of an equilibrium without valued currency and with pooling markets for IOUs. In this equilibrium, despite the possibility of doing so, entrepreneurs do not choose to issue enough IOUs to entirely relax their enforcement constraints. This follows from an existing tension between the enforcement constraints and the incentives to sell capital under asymmetric information. On one hand, selling a marginal unit of capital under asymmetric information is costly to the entrepreneur because he receives a pooling price for an object that he values above that price. On the other hand, when financial frictions are active, they provide the incentives that support trade under asymmetric information because relaxing these constraints is valued by the entrepreneur. When constraints are entirely relaxed by acquiring sufficient funds, liquidity has no value on the margin because there is no point in having additional funds. Nevertheless, to obtain this amount, the entrepreneur must incur a loss from selling a marginal asset in a pooling market. This result is explained in Bigio [2010].

Thus, the takeaway lesson is that financial frictions must be active in order to support trade under asymmetric information. In other words, when liquidity is needed to enforce efficient employment or investment, the economy will feature under-employment and under-investment. This result also explains a a similarity between this economy and a cash-in-advance economy. In particular, real balances of fiat money may never be sufficient to relax the enforcement constraints with probability 1 in future states because, in that case, fiat money becomes a dominated assets. We summarize this result

Theorem 8 (Inefficiency)**.** *In any equilibrium, with or without valued currency, employment at the firm is always sub-efficient infinitely often if and only if* $\theta < (1-\alpha)$*.*

The intuition behind this result is that if $\theta < (1-\alpha)$, producers cannot credibly pledge workers the labor share of output unless they use liquid funds. Thus, in order to attain efficient employment, liquidity is needed. In contrast, if the liquidity is sufficient to relax this constraint in all states, fiat money becomes a dominated assets. Hence, fiat money will not be valued in an equilibrium where constraints never bind. In contrast, if constraints never bind, there's no issuance of IOUs either.

This result also highlights the tension between real fiat money balances and real balances of IOUs. As explained before, the incentives to issue IOUs are weekend when entrepreneur's hold more real balances.

2.5 Effects of Dispersion Shocks

A version without fiat money. This example is calibrated setting $M_t = 0$. The example shows the effects of reductions in liquidity. It shows up as a labor wedge.

3 Experiments with Fiat Money and IOUs

In this section, I avoid referring to any time subscript.

3.1 Effects of Dispersion and Fiat money Shocks

Effects of ξ. We can use the model to studye the effects of shocks to dispersion.

Effects of μ*.* Notice that a once and for all shock to fiat money has no effects for the equilibrium. We can use the model, however, to study how different monetary rules can improve outcomes. It would be interesting to know if there are interesting trade-offs. *What are the differences between expected changes in interest rates and unexpected changes?* Can we contrast *Fisher vs. Liquidity effects* on interest rates?

Nominal Wage Rigidities. It is easy to introduce a crude form of nominal rigidities. In particular, the environment remains the same except that we can assume that wages and prices of goods cannot fall by more than a fraction η. This crude rigidity is also studied in **?**. The relevant state variable here then is

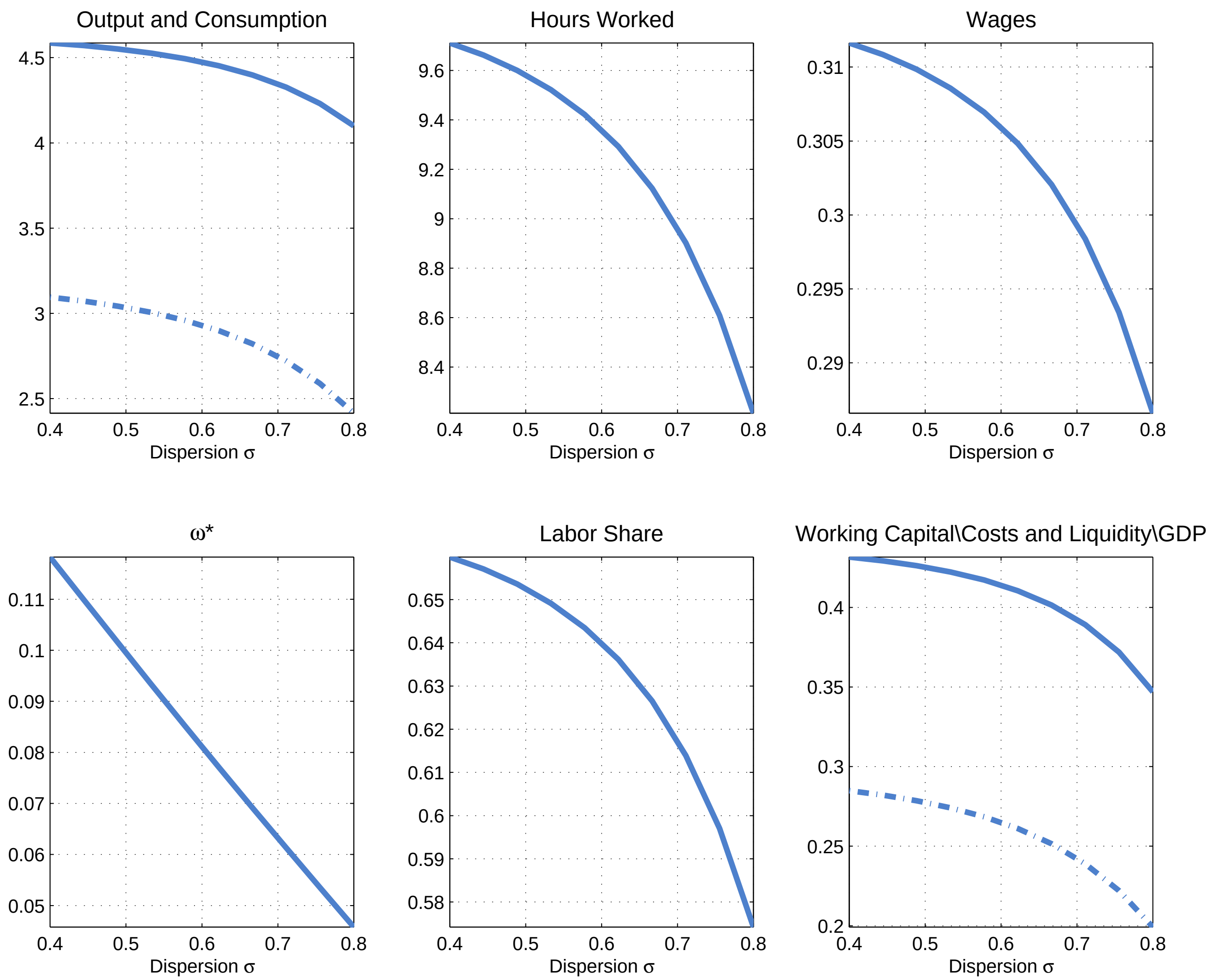

Figure 1: **Pooling equilibrium Without Fiat Money.** The figure presents some equilibrium variables as function of ϕ which captures the degree of asymmetric information in this economy.

$X_t = (\mu_t, \xi_t, w_{t-1})$. The purpose of this extension is to introduce a form of non-neutrality of fiat money injections into this economy in a way that keeps that preserves the tractability.

Phillips curve. One question is whether there is a phillips curve in the environment. I suspect yes. As liquidity falls, employment falls and there is a price appreciation —money is more valuable. Is this an exploitable Phillips curve?

References

Saki Bigio. Financial risk capacity. 2010. URL "http://homepages.nyu.edu/ msb405/".

Lawrence J. Christiano, Martin Eichenbaum, and Charles L. Evans. Nominal rigidities and the dynamic effects of a shock to monetary policy. *The Journal of Political Economy*, 113(1):pp. 1–45, 2005. URL http://www.jstor.org/stable/3555296.

Urban Jermann and Vincenzo Quadrini. Macroeconomic effects of financial shocks. 2009. URL http://finance.wharton.upenn.edu/ jermann/creditshocks-aug10u.pdf.

Nobuhiro Kiyotaki and John Moore. Evil is the root of all money. *The American Economic Review*, 92(2): 62–66, May 2002.

Nobuhiro Kiyotaki and John Moore. Liquidity, business cycles, and monetary policy. 2008. URL http://www.princeton.edu/ kiyotaki/papers/ChiKM6-1.pdf.

Guillaume Rocheteau. A monetary approach to asset liquidity. 2009. URL http://www.grocheteau.com/wpapers/MKPI-121008.pdf.

4 Proofs

4.1 Proof of Lemma 3

The proof in this section follows the one in Bigio 2011. For the proof, $w\left(s^t\right)$ is referred simply as w. Rearranging the incentive compatibility constraints this problem consists of solving:

$$\begin{aligned} \pi\left(x, X\right) &= \max_{l \geq 0, \sigma \in (0,1)} Al^{1-\alpha} - wl \text{ subject to} \\ \sigma wl &\leq \theta^L Al^{1-\alpha} \text{ and } \left(1-\sigma\right) wl \leq r. \end{aligned}$$

Denote the solutions to this problem by $\left(l^*, \sigma^*\right)$. The unconstrained labor demand is $l^{unc} \equiv \left[\frac{A(1-\alpha)}{w}\right]^{\frac{1}{\alpha}}$. A simple manipulation of the constraints yields a pair of equations that characterize the constraint set:

$$\begin{aligned} l &\leq \left[A \frac{\theta^L}{\sigma w}\right]^{\frac{1}{\alpha}} \equiv l^1\left(\sigma\right) &(2) \\ l &\leq \frac{r}{\left(1-\sigma\right) w} \equiv l^2\left(\sigma\right) &(3) \\ \sigma &\in (0,1) \end{aligned}$$

As long as l^{unc} is not a part of the constraint set, at least one of the constraints will be active because the objective is increasing in l for $l \leq l^{unc}$. In particular, the tighter constraint will bind as long as $l \leq l^{unc}$. Thus, $l^* = \min\left\{l^1\left(\sigma^*\right), l^2\left(\sigma^*\right)\right\}$ if $\min\left\{l^1\left(\sigma^*\right), l^2\left(\sigma^*\right)\right\} \leq l^{unc}$ and $l^* = l^{unc}$ otherwise. Therefore, note that (2) and (3) impose a cap on l depending on the choice of σ. Hence, in order to solve for l^*, we need to know σ^*. First observe that (2) is a decreasing function of σ. The following properties can be verified immediately:

$$\lim_{\sigma \to 0} l^1\left(\sigma\right) = \infty \text{ and } l^1\left(1\right) = \left(\frac{\theta^L}{\left(1-\alpha\right)}\right)^{\frac{1}{\alpha}} \left[\frac{A}{w}\left(1-\alpha\right)\right]^{\frac{1}{\alpha}} = \left(\frac{\theta^L}{\left(1-\alpha\right)}\right)^{\frac{1}{\alpha}} l^{unc} \qquad (4)$$

The second constraint curve (3) presents the opposite behavior. It is increasing and has the following limits,

$$l^2\left(0\right) = \frac{r}{\omega} \text{ and } \lim_{\sigma \to 1} l^2\left(\sigma\right) = \infty$$

These properties imply that $l^1\left(\sigma\right)$ and $l^2\left(\sigma\right)$ will cross at most once if $x > 0$. Because the objective is independent of σ, the entrepreneur is free to choose σ in a to relax the constraint on l as much as possible. Since $l^1\left(\sigma\right)$ is decreasing and $l^2\left(\sigma\right)$ increasing, the optimal choice of σ^* solves $l^1\left(\sigma^*\right) = l^2\left(\sigma^*\right)$ so that l is as large as possible. This implies that both constraints will bind if one of them must bind. Adding them up, we find that $l^{cons}(x)$ is the largest solution to

$$\theta^L A l^{1-\alpha} - wl = -r \tag{5}$$

This equation defines $l^{cons}(x)$ as the largest solution of this implicit function: when $r = 0$, this function has two zeros so the restriction to the largest root prevents us from picking $l = 0$. Indeed, if $r = 0$, then $\sigma = 1$, does not imply that l should be 0.

Thus, we have that,

$$l^*(r) = \min\left\{l^{cons}(r), l^{unc}\right\}$$

Since $l^1(\sigma)$ is monotone decreasing, if $\theta^L \geq (1-\alpha)$, then, $l^1(1) \geq l^{unc}$, by (4). Because for $x > 0$, $l^1(\sigma)$ and $l^2(\sigma)$ cross at some $\sigma < 1$, then, $l^{cons} > l^{unc}$ and $l^* = l^{unc}$. Moreover, if $r = 0$, then the only possibility implied by the constraints of the problem is to set $\sigma = 1$. But since, $l^1(1) \geq l^{unc}$, then $l^* = l^{unc}$. Thus, we have shown that $\theta^L \geq (1-\alpha)$ is sufficient to guarantee that labor is optimally chosen regardless of the value of r. This proves the second claim in the proposition.

Assume now that $l^{unc} \leq \frac{r}{\omega}$. Then, the wage bill corresponding to the efficient employment can be guaranteed upfront by the entrepreneur. Obviously, $r \geq wl^{unc}$ is sufficient for optimal employment.

To pin down the necessary condition for the constraint to bind, observe that the profit function in (5) is concave with an positive interior maximum. Thus, at $l^{cons}(r)$, the left hand side of (5) is decreasing. Therefore, if $l^{cons}(r) < l^{unc}$, then it should be the case that $\theta^L A \left(l^{unc}\right)^{1-\alpha} - wl^{unc} < -r$. Substituting the formula for l^{unc} yields the necessary condition for the constraints to be binding:

$$r < w^{1-\frac{1}{\alpha}} \left[A(1-\alpha)\right]^{\frac{1}{\alpha}} \left(1 - \frac{\theta^L}{(1-\alpha)}\right)$$

This shows that if $\frac{\theta^L}{(1-\alpha)} < 1$, at least some liquidity is needed to achieve efficient employment.

4.2 Marginal Value of Liquidity

Here I derive an expression for π_r. Note that

$$\Pi_r = \frac{\partial \Pi}{\partial l^*} \frac{\partial l^*}{\partial r}.$$

The marginal return on labor is $\frac{\partial \Pi}{\partial l^*} = (1-\alpha) A l^{-\alpha} - w$ whereas the marginal increase in labor given an extra unit of liquid funds is given by an aplication of the implicit function theorem. Define:

$$G\left(l,r\right)=\theta^{L}Al^{1-\alpha}-wl+r$$

then,

$$\frac{\partial l^{*}}{\partial r}=-\frac{G_{r}\left(l,r\right)}{G_{l}\left(l,r\right)}=-\frac{1}{\left(1-\alpha\right)\theta^{L}Al^{-\alpha}-w}.$$

Combining this result with the expression for $\frac{\partial\Pi}{\partial l^{*}}$ obtains,

$$\Pi_{r}=-\frac{\left(1-\alpha\right)Al^{-\alpha}-w}{\left(1-\alpha\right)\theta^{L}Al^{-\alpha}-w}.$$

4.3 Proof of Lemma 4

4.4 Proof of Proposition 7

The strategy of the proof is guess and verify. Let $W\left(z,s^{t}\right)$ be the entrepreneur's virtual wealth (corresponding to the definition under either separating or pooling market equilibria) where $W\left(z,s^{t}\right)=\pi(r_{t}(z,s^{t}),w(s^{t}))+q_{t}\left(s^{t}\right)k_{t}(z,s^{t-1})+p^{-1}(s^{t})m_{t}(z,s^{t})$ and where $r_{t}(z,s^{t})$ is the available liquid funds for the entrepreneur. The guess is that policy functions are linear in virtual wealth: $c_{t}\left(z,s^{t}\right)=\left(1-\beta\right)W_{t}\left(z,s^{t}\right)$, $k_{t+1}\left(z,s^{t}\right)=\rho\left(z,s^{t}\right)\beta W_{t}\left(z,s^{t}\right)$, and $\tilde{m}_{t+1}\left(z,s^{t}\right)=\left(1-\rho\left(z,s^{t}\right)\right)\left(1-\beta\right)W_{t}\left(z,s^{t}\right)$. The exact weight $\rho\left(z,s^{t}\right)$ are functions of the state.

Recall that the entrepreneur must solve:

$$\max_{c_{t}(z,s^{t}),x_{t}(s^{t}),\tilde{m}_{t+1}(s^{t})}\mathbb{E}_{0}\left[\sum_{t\geq0}\beta^{t}U(c(z,s^{t}))\right]$$

subject to:

$$\begin{aligned}
c_{t}(z,s^{t})+q(s^{t})x_{t}(z,s^{t})+p^{-1}(s^{t})\tilde{m}_{t+1}(z,s^{t})&=[\pi(r_{t}(z,s^{t}),w(s^{t}))+r_{t}(z,s^{t})]k_{t}(z,s^{t-1})\\
\tilde{m}_{t+1}(z,s^{t+1})&=\left(1+\mu\left(s^{t+1}\right)\right)\tilde{m}_{t+1}(z,s^{t})\\
r_{t}(z,s^{t})&=\frac{p^{-1}(s^{t})m_{t}(z,s^{t})+q\left(s^{t}\right)v_{t}(z,s^{t})k_{t}(z,s^{t-1})}{k_{t}(z,s^{t-1})}\\
k_{t+1}(z,s^{t})&=x_{t}(z,s^{t})+k_{t}(z,s^{t-1})[1-v_{t}(z,s^{t})]
\end{aligned}$$

where $\pi(r,w)$ is the value of Problem 2. Substituting in the capital accumulation equation to get rid of

$x_t(z, s^t)$, the budget constraint is:

$$\begin{aligned} & c_t(z,s^t) + q(s^t)k_{t+1}(z,s^t) + p^{-1}(s^t)\tilde{m}_{t+1}(z,s^t) \\ = & \left[\pi(r_t(z,s^t), w(s^t)) + q_t\left(s^t\right)\right] k_t(z,s^{t-1}) + p^{-1}(s^t)m_t(z,s^t) \end{aligned}$$

Now assign to each budget constraint for each possible history a Lagrange multiplier $\lambda\left(s^t\right)$. The entrepreneur's problem becomes:

$$\begin{aligned} & \max_{c_t(z,s^t), k_{t+1}(s^t)\geq 0, m_{t+1}(s^t)\geq 0} \mathbb{E}_0 \left[\sum_{t\geq 0} \beta^t U(c(z,s^t))\right] \\ & +\lambda\left(s^t\right)\left[\left[\pi(r_t(z,s^t), w(s^t)) + q_t\left(s^t\right)\right] k_t(z,s^{t-1}) + p^{-1}(s^t)m_t(z,s^t)\right. \\ & \left. -c_t(z,s^t) - q(s^t)k_{t+1}(z,s^t) - p^{-1}(s^t)\tilde{m}_{t+1}(z,s^t)\right]. \end{aligned}$$

Note that the proposed guess satisfies the consolidated budget constraint,

$$(1-\beta) W_t\left(z,s^t\right) + \rho\left(z,s^t\right)\beta W_t\left(z,s^t\right) + \left(1-\rho\left(z,s^t\right)\right)\beta W_t\left(z,s^t\right) = W_t\left(z,s^t\right),$$

so the guess satisfies the budget contraint.

The first order conditions for this problem are:

$$\begin{aligned} c_t(z,s^t) &: \frac{\beta^t}{c_t(z,s^t)} = \lambda\left(s^t\right) \\ k_{t+1}(z,s^t) &: \lambda\left(s^t\right) q(s^t) = \lambda\left(s^{t+1}\right)\left[\pi(r_t(z,s^t), w(s^t))\right. \\ & \left. +q\left(s^t\right) + k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}), w(s^t))\frac{\partial \pi_r(r_{t+1}(z,s^t), w(s^{t+1}))}{\partial k_{t+1}(z,s^t)}\right] \\ & +\gamma\left(z,s^t\right) \\ \tilde{m}_{t+1}(z,s^t) &: \lambda\left(s^t\right) p^{-1}(s^t) = \lambda\left(s^{t+1}\right)\left[\left(1+\mu\left(s^{t+1}\right)\right) p^{-1}(s^{t+1})\right. \\ & +k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}), w(s^t))\frac{\partial \pi_r(r_{t+1}(z,s^t), w(s^{t+1}))}{\partial \tilde{m}_{t+1}(z,s^t)} \\ & +\eta_t\left(z,s^t\right) \end{aligned}$$

Substituting $\lambda\left(s^t\right)$ from the first F.O.C. and summing up across states weighting states by $\Pi\left(s^{t+1}|s^t\right)$, we obtain:

$$\begin{aligned} q(s^t) &= \mathbb{E}_t[\beta \frac{c_t(z,s^t)}{c_{t+1}(z,s^{t+1})}[\pi(r_t(z,s^t),w(s^t)) + q_t\left(s^{t+1}\right) \\ &\quad + k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{\partial \pi_r(r_{t+1}(z,s^t),w(s^{t+1}))}{\partial k_{t+1}(z,s^t)}]] + \gamma\left(s^t\right) \\ p^{-1}(s^t) &= \mathbb{E}_t[\beta \frac{c_t(z,s^t)}{c_{t+1}(z,s^{t+1})}[\left(1+\mu\left(s^{t+1}\right)\right)p^{-1}(s^{t+1}) \\ &\quad + k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{\partial \pi_r(r_{t+1}(z,s^t),w(s^{t+1}))}{\partial \tilde{m}_{t+1}(z,s^t)}]] + \eta_t\left(s^t\right) \end{aligned}$$

where $\gamma\left(s^t\right)$ and $\eta_t\left(s^t\right)$ are the corresponding Kuhn-Tucker multipliers associated with the boundaries of the portfolio problem. It is convenient now to use the policy functions of our initial guess. When this is the case, note that stochastic discount factor becomes:

$$\begin{aligned} &\beta\left[\frac{(1-\beta)W_t\left(s^t\right)}{(1-\beta)W_{t+1}\left(s^{t+1}\right)}\right] \\ =& \left[\frac{\beta(1-\beta)W_t\left(s^t\right)}{(1-\beta)\left[\frac{(\pi(r_{t+1}(z,s^{t+1}),w(s^{t+1}))+q(s^{t+1}))}{q(s^t)}\rho\left(s^{t+1}\right)+\left(1+\mu\left(s^{t+1}\right)\right)\frac{p^{-1}(s^{t+1})}{p^{-1}(s^t)}\left(1-\rho\left(s^{t+1}\right)\right)\right]W_t\left(s^t\right)}\right] \end{aligned}$$

and defining,

$$\begin{aligned} R^m\left(s^{t+1}\right) &\equiv \left(1+\mu\left(s^{t+1}\right)\right)\frac{p\left(s^t\right)}{p\left(s^{t+1}\right)} \\ R^k\left(\rho,s^{t+1}\right) &\equiv \frac{q\left(s^{t+1}\right)+\pi\left(r\left(\rho,s^{t+1}\right),w\left(s^t\right)\right)}{q\left(s^t\right)}. \end{aligned}$$

the stochastic discount factor becomes:

$$\left[\frac{\beta}{\left[R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right)\rho\left(s^{t+1}\right)+R^m\left(s^{t+1}\right)\left(1-\rho\left(s^{t+1}\right)\right)\right]}\right].$$

Note also that for the proposed policy functions,

$$r_{t+1}\left(s^{t+1}\right) = q\left(s^t\right)\left(1+\mu\left(s^{t+1}\right)\right)\frac{p^{-1}(s^{t+1})}{p^{-1}(s^t)}\frac{(1-\rho\left(z,s^t\right))}{\rho\left(z,s^t\right)} + q\left(s^{t+1}\right)v_{t+1}(z,s^{t+1}).$$

Under this guess we have,

$$
\begin{aligned}
\frac{\partial r_{t+1}(z,s^t)}{\partial k_{t+1}(z,s^t)} &= -\frac{p^{-1}(s^{t+1})\left(1+\mu\left(s^{t+1}\right)\right)\tilde{m}_{t+1}(z,s^t)}{k_{t+1}(z,s^t)^2} \rightarrow \\
k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{\partial r_{t+1}(z,s^t)}{\partial k_{t+1}(z,s^t)} &= -q\left(s^t\right)\pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{p^{-1}(s^{t+1})\left(1-\rho_{t+1}(z,s^t)\right)}{p^{-1}(s^t)\rho_{t+1}(z,s^t)}
\end{aligned}
$$

where the second line uses the guessed policies.

Similarly,

$$
\begin{aligned}
\frac{\partial r_{t+1}(z,s^t)}{\partial m_{t+1}(z,s^t)} &= \frac{p^{-1}(s^{t+1})}{k_{t+1}(z,s^t)}\left(1+\mu\left(s^{t+1}\right)\right) \rightarrow \\
k_{t+1}(z,s^t)\pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{\partial r_{t+1}(z,s^t)}{\partial m_{t+1}(z,s^t)} &= \pi_r(r_{t+1}(z,s^{t+1}),w(s^t))p^{-1}(s^{t+1})\left(1+\mu\left(s^{t+1}\right)\right)
\end{aligned}
$$

Combining these calculations, with the expression for the stochastic discount factor delivers,

$$
\frac{1}{\beta} = \mathbb{E}_t\left[\frac{R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right) - \pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{p^{-1}(s^{t+1})\left(1-\rho_{t+1}(z,s^t)\right)}{p^{-1}(s^t)\rho_{t+1}(z,s^t)}\left(1+\mu\left(s^{t+1}\right)\right) + \gamma\left(s^t\right)/q\left(s^t\right)}{\left[R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right)\rho\left(s^{t+1}\right) + R^m\left(s^{t+1}\right)\left(1-\rho\left(s^{t+1}\right)\right)\right]}\right]
$$

and,

$$
\frac{1}{\beta} = \mathbb{E}_t\left[\frac{R^m\left(s^{t+1}\right) + \pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{p^{-1}(s^{t+1})}{p^{-1}(s^t)}\left(1+\mu\left(s^{t+1}\right)\right) + \eta_t\left(s^t\right)p^{-1}\left(s^t\right)}{\left[R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right)\rho\left(s^{t+1}\right) + R^m\left(s^{t+1}\right)\left(1-\rho\left(s^{t+1}\right)\right)\right]}\right].
$$

Combining both equations we obtain the following condition:

$$
\begin{aligned}
&\mathbb{E}_t\left[\frac{R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right) - \pi_r(r_{t+1}(z,s^{t+1}),w(s^t))\frac{p^{-1}(s^{t+1})}{p^{-1}(s^t)\rho_{t+1}(z,s^t)}\left(1+\mu\left(s^{t+1}\right)\right) - \eta_t\left(s^t\right)p^{-1}\left(s^t\right) + \gamma\left(s^t\right)/q\left(s^t\right)}{\left[R^k\left(\rho\left(s^{t+1}\right),s^{t+1}\right)\rho\left(s^{t+1}\right) + R^m\left(s^{t+1}\right)\left(1-\rho\left(s^{t+1}\right)\right)\right]}\right] \\
&= 0.
\end{aligned}
$$

Finally, observe that $\eta_t\left(s^t\right) > 0$ if and only if $\tilde{m}_{t+1}(z,s^t) \geq 0$ is binding and $\gamma\left(s^t\right) > 0$ if and only if $k_{t+1}(z,s^t) \geq 0$ is binding. Since the Inada conditions hold, agents will always hold positive balances of either $\tilde{m}_{t+1}$ or k_{t+1}, so either $\eta_t\left(s^t\right)$ or $\gamma\left(s^t\right)$ will be greater than zero but not both.

Now consider the following portfolio problem:

$$\Omega\left(v,s^{t}\right) = \max_{\rho\in[0,1]} \mathbb{E}_{t}\left[\log\left(\rho R^{k}\left(\rho,s^{t+1}\right)+\left(1-\rho\right)R^{m}\left(s^{t+1}\right)\right)|s^{t}\right]$$

$$s.t.$$

$$r\left(\rho,s^{t+1}\right) = -q\left(s^{t}\right)\left(1+\mu\left(s^{t+1}\right)\right)\frac{p^{-1}(s^{t+1})}{p^{-1}(s^{t})}\frac{(1-\rho)}{\rho}+q\left(s^{t+1}\right)v$$

The first order conditions of this problem are:

$$\mathbb{E}_{t}\left[\frac{R^{k}\left(\rho,s^{t+1}\right)+\rho\pi_{r}\left(r\left(\rho,s^{t+1}\right),w\left(s^{t}\right)\right)\frac{\partial r\left(\rho,s^{t+1}\right)}{\partial\rho}-R^{m}\left(s^{t+1}\right)}{\rho R^{k}\left(\rho,s^{t+1}\right)+\left(1-\rho\right)R^{m}\left(s^{t+1}\right)}\right]=\varkappa\left(s^{t}\right)$$

where

$$\varkappa\left(s^{t}\right)=\begin{cases}\varkappa\left(s^{t}\right)>0 \text{ if } \frac{R^{k}\left(1,s^{t+1}\right)+\pi_{r}\left(r\left(1,s^{t+1}\right),w\left(s^{t}\right)\right)\frac{\partial r\left(\rho,s^{t+1}\right)}{\partial\rho}|_{\rho=1}-R^{m}\left(s^{t+1}\right)}{R^{k}\left(1,s^{t+1}\right)}>0\\ \varkappa\left(s^{t}\right)<0 \text{ if } \frac{R^{k}\left(0,s^{t+1}\right)+\pi_{r}\left(r\left(0,s^{t+1}\right),w\left(s^{t}\right)\right)\frac{\partial r\left(\rho,s^{t+1}\right)}{\partial\rho}|_{\rho=0}-R^{m}\left(s^{t+1}\right)}{R^{m}\left(s^{t+1}\right)}<0\\ \varkappa\left(s^{t}\right)=0 \text{ otherwise}\end{cases}$$

Now since,

$$\frac{\partial r\left(\rho,s^{t+1}\right)}{\partial\rho}=-q\left(s^{t}\right)\left(1+\mu\left(s^{t+1}\right)\right)\frac{p^{-1}(s^{t+1})}{p^{-1}(s^{t})}\frac{1}{\rho^{2}}$$

and this implies:

$$\mathbb{E}_{t}\left[\frac{R^{k}\left(\rho,s^{t+1}\right)-\pi_{r}\left(r\left(\rho,s^{t+1}\right),w\left(s^{t}\right)\right)\left(1+\mu\left(s^{t+1}\right)\right)\frac{p^{-1}(s^{t+1})}{p^{-1}(s^{t})}\frac{1}{\rho}-R^{m}\left(s^{t+1}\right)}{\rho R^{k}\left(\rho,s^{t+1}\right)+\left(1-\rho\right)R^{m}\left(s^{t+1}\right)}\right]=\varkappa\left(s^{t}\right).$$

Note that the first order condition of this portfolio problem is identical to the first order conditions obtained from the entrepreneur's problem. Hence, any ρ^{*} solution to $\Omega\left(v,s^{t}\right)$, is also optimal for the entrepreneur's choice of $\rho(z,s^{t})$. Since the portfolio weight is optimal, and our guess satisfies the first order condition for the entrepreneur's problem, this verifies that the guessed choice satisfies the necessary first order conditions of this problem.

5 Conditions for Interior Solutions of ρ

6 Equilibrium Equations for model without fiat money

Equilibria without fiat currency is characterized by the following set of conditions:

$$
\begin{aligned}
q &= \beta W \\
W &= (\Pi(r) + q) \\
\Pi(r) &= l(r)^{(1-\alpha)} - wr \\
\Pi_r &= -\frac{(1-\alpha) l^{1-\alpha} - wl}{\theta(1-\alpha) l^{1-\alpha} - wl} \\
r &= p \int_0^{\omega^*} f_\phi(\omega) \, d\omega \text{ for the pooling equilibria} \\
p &= qE[\lambda(\omega) | \omega < \omega^*] \;\; \text{for the pooling equilibria} \\
r &= \int_0^1 \tilde{q}(\omega) \theta(\omega) \, d\omega \text{ for the separating equilibria} \\
r &= m + q\nu \\
\lambda(\omega^*) &= (1 + \Pi_r) E[\lambda(\omega) | \omega < \omega^*] \\
\nu &= E[\lambda(\omega) | \omega < \omega^*] F(\omega^*) \\
\theta l^{(1-\alpha)} - wl &= -r \\
w &= (l^s)^\nu
\end{aligned}
$$

This system is parameterized by a single equation in ω^*. Solve for $w\left(\nu^{pool}\right)$ by solving the last two equations as functions of ν. Then, define $\Pi(n)$ and $\Pi_r(n)$ implicitly via $l(n)$. Then, $W(n)$ and $q(n)$ are defined implicitly. Finally, find the value of $\lambda(\omega^*)$. Invert this to obtain the new n. We look for a fixed point.

$$\theta l^{(1-\alpha)} - (l)^{1/\nu+1} = -n$$

Always exists a solution. Minimal liquidity. Plug in $l(n)$.

$$\Pi(n) = l(n)^{(1-\alpha)} - l(n)^{1/\nu+1}$$

Hence, $q(n)$ is given by:

$$q(n) = \beta \left[l(n)^{(1-\alpha)} - l(n)^{1/\nu+1} + n \right] > 0$$

and

$$\Pi_r(n) = \frac{(1-\alpha) l(n)^{(1-\alpha)} - l(n)^{1/\nu+1}}{(1-\alpha)\theta l(n)^{(1-\alpha)} - l(n)^{1/\nu+1}}$$

Then,

$$\lambda(\omega^*) = \beta \Pi_r(n) q(n) E[\lambda(\omega) | \omega < \omega^*]$$

To argue that there's a unique solution. Find $\Pi_{r,n}(n)$ and $q(n)$ from equaiton above, and use the assumption on $E[\lambda(\omega)|\omega < \omega^*] / \lambda(\omega^*)$. With this, one can argue monotonicity of n. Proceed in the following way: $q(n) = \frac{\beta \Pi(n)}{(1-\beta)}$. $W(n) = \frac{\beta}{(1-\beta)} \Pi(n)$.

6.1 Equilibria with fiat money.

To show neutrality, use H(0) argument. Homogeneity of degree zero. Use $m_{t+1} = \mu_t \tilde{m}_{t+1}$. Meaning that money authority pays a return to agents. For this reason, actual return is: $R^m(\phi) = \frac{p_t + \pi_r(r,w)}{p_{t+1}\mu_t}$. Check this thing.

6.1.1 Non-Stochastic Steady State.

Fix some $\xi_t = \bar{\xi}$ constant accross time. Then, let m^* be the real balances in fiat currency. Equilibria with fiat currency is characterized by the following set of conditions:

$$
\begin{aligned}
q &= \rho\beta W \\
m^* &= (1-\rho)\,\beta W \\
W &= (\Pi + q + m^*) \\
\Pi &= l^{(1-\alpha)} - wl \\
\Pi_r &= \frac{(1-\alpha)\,l^{-\alpha} - wl}{(1-\alpha)\,\theta l^{-\alpha} - wl} \\
\nu &= qE_\phi\left[\lambda(\omega)\,|\omega < \omega^*\right] F(\omega^*) \\
\lambda(\omega^*) &= (1+\Pi_r)\,E_\phi\left[\lambda(\omega)\,|\omega < \omega^*\right] \\
\theta l^{(1-\alpha)} - wl &= -(n + m^*) \\
w &= (l^s)^{1/\nu}
\end{aligned}
$$

However, agents don't solve a portfolio problem but must have a return indifference condition satisfied by ρ :

$$
R^k(\rho) - R^m = -\rho R'^k(\rho)
$$

This equation states that the excess returns should equal the liquidity premia. Using formulas derived earlier, we obtain:

$$
\frac{\Pi}{q} = \rho\frac{\Pi_r}{q}\frac{\partial r}{\partial \rho} = \frac{\Pi_r}{\rho} \to \rho = \frac{q\Pi_r}{\Pi}.
$$

Using this expression we have:

$$
\bar{W} = \frac{\beta}{(1-\beta)}\Pi
$$

$$
m^* = \left(1 - \frac{q\Pi_r}{\Pi}\right)\beta W
$$

and,

$$
m^* + q = W.
$$

Clearing this equation yields:

$$\begin{aligned}\Pi W - \Pi q &= (\beta W - q\beta W \Pi_r) \rightarrow \\ q &= \frac{(\Pi - \beta) W}{(\Pi - \beta W \Pi_r)}\end{aligned}$$

6.1.2 Stochastic Version

Let m^* be the real balances in fiat currency. Equilibria with fiat currency is characterized by the following set of conditions:

$$\begin{aligned}
q &= \rho\beta W \\
m^* &= (1-\rho)\beta W \\
W &= (\Pi + q + m^*) \\
\Pi &= l^{(1-\alpha)} - wl \\
\Pi_r &= \frac{(1-\alpha) l^{-\alpha} - wl}{(1-\alpha)\theta l^{-\alpha} - wl} \\
n &= q E_\phi [\lambda(\omega) | \omega < \omega^*] \\
\lambda(\omega^*) &= (1+\Pi_r) E_\phi [\lambda(\omega) | \omega < \omega^*] \\
\theta l^{(1-\alpha)} - wl &= -(n + m^*) \\
w &= (l^s)^{1/\nu} \\
\rho &= \arg\max_{\rho^*} \left[\log\left(\rho^* R^k(\phi', \rho^*) + (1-\rho^*) R^m(\phi')\right) | \phi\right] \\
R^m(\phi|\phi) &= 1 \\
R^k(\phi|\phi, \rho) &= 1 + \frac{\Pi(r, w)}{q}
\end{aligned}$$

System can be solved via two iterative steps. First we solve for a first block of this system is parameterized by a single equation in ω^* given some amount of liquidity n. Solve for $w(n)$ by solving the last two equations as functions of ν. Then, define $\Pi(n)$ and $\Pi_r(n)$ implicitly via $l(n)$ using equation xxx. xxx. xxx. Then, $W(n, \rho)$, $m^*(n, \rho)$ and $q(n, \rho)$ are solved by defining them implicitly via the first three equations. Then, using the solutions for each possible state, find the corresponding values of $R^m(\phi)$ and $R^k(\phi'|\phi)$ and uses them to compute solutions to ρ. Follow this step until convergence. Once this is done, update liquidity value.

Updating ρ. Sum (1) and (2) to obtain, $W = (\Pi + \beta W) \rightarrow W = \frac{\Pi}{(1-\beta)} \rightarrow q = \rho\frac{\beta\Pi}{(1-\beta)}$ and $m^* =$

$(1-\rho)\frac{\beta\Pi}{(1-\beta)}$. We update the return matrixes for $R^m(\phi'|\phi)$ and $R^k(\phi'|\phi)$ and solve

$$\rho = \arg\max_{\rho^*} \mathbb{E}\left[\log\left(\rho^* R^k\left(\phi'\right) + \left(1-\rho^*\right) R^m\left(\phi'\right)\right)|\phi\right].$$