Homework #3

—

UCLA - Fall, 2017

ECON 221C MONETARY ECON III

Monetary Economics in continuous time.

Saki Bigio

—

**Excercise 1 [Stochastic Calculus]. In order to do this question, you will want to read Chapters 2-5 of Evan's Introduction to SDE's or Chapters 2-3 of Stokey's book. You may want to consult Oksendahl or Harris.**

Use Ito's Lemma to show or solve:

1. Calculation of integrals:

$$\int_0^T W^N dW = \frac{1}{N} W^N(T) - \frac{N}{2} \int_0^T W dt.$$

2. Prove that the even momenta satisfy:

$$\mathbb{E}\left[W^{2k}(t)\right] = \frac{(2k)!t^k}{2^k k!} \text{ for } k \text{ integer}$$

3. Show that:

$$\mathbb{E}\left[W^2(t)\right] = \frac{t^3}{3}$$

4. Show that:

$$X_t = \frac{B_t}{1-t} \text{ solves } dX_t = -\frac{X_t}{1+t}dt + \frac{1}{1+t}dB_t$$

5. Solve the OH process

$$dX_t = -\mu X_t dt + \sigma dB_t.$$

For that, first work with $\mu = -1$. Apply Ito's Lemma to $d(e^{-t}X_t)$ and obtain an expression for $X_t$ as a function of time, $X_0$ and a stochastic integral. Compute the first to moments of $X_t$. Argue that $X_t$ is normally distributed based on the properties of stochastic integrals. Provide a distribution for $X_t$.

6. Use Ito's Lemma to solve for $X_t$ in:

$$dX_t = (\alpha - \mu X_t)\, dt + \sigma dB_t.$$

Do that in one line by invoking your solution to 5.

7. Let $P_t^i$ be the price of good $i$ at time $t$. Let $P_t^i$ follow a geometric Brownian motion with Brownian term $B_t^i$. Each shock is i.i.d. Consider a utility function of the CES class:

$$U(\hat{x}) \equiv \left( \sum_{i \in I} \alpha_i^{1/\theta} x_i^{1-1/\theta} \right)^{\frac{\theta}{\theta-1}}$$

Construct the indirect utility function corresponding to the budget constraint:

$$< P \cdot x_i >= e.$$

In order to perfom the following question, you must read about the multidimentional version of Ito's Lemma.

Represent $U(\hat{x})$ now as a function of consumption and a preference shock vector that depends on $P_t^i$.

(a) Provide a diffussion proccess for that preference shock.

(b) Describe the behavior of that process a the limit where $N \to \infty$ —i.e. as we add goods, while keeping $\sum \alpha_i = 1$.

(c) As t goes to infinity, where does the preference shock converge to? How does your answer depend on $\theta$?

(d) Assume now that instead of e, now the country is endowed (with constant endowments) with a subset J of the I possible goods. Rewrite the Budget constraint. Solve for the terms of trade. Solve for the Real exchange rate. Write a stochastic process for the terms of trade. Write a stochastic process for J.

(e) Assume now that a subset Z of J are non-tradeable goods. Thus, there prices is determined endogenously. Find the terms of trade, the real exchange rate and their corresponding stochastic processes.

**Excercise 2 [HJB Equation - Stopping Time].** Assume U is consistent with CRRA utilty. Assume that $X_t$ follows a Geometric Brownian Motion with positive drift. Assume that the agent gets 0 utilty from $t \in [0, T)$. However, the agent gets utility $U(X_T)$ at date $T$. The value function in this problem is given by some $V(X, t)$ assuming a discount factor $\rho$.

(a) **Direct Approach:** use the solution to the GBM to solve for the value of the agent at time $T$. For this, simply compute the expectation of $U(X_T)$, conditional on the value of $X(t)$ and $t$. Given the value of $X(t)$ and the time remaining $T - t$, you can solve for the distribution of $X(t)$ or you can integrate the objective (to get the expectation) by use of the distribution of the underlying Brownian motion ($dW_t$).

(b) **Indirect Approach:** Derive the HJB Equation:

$$\rho V(X, t) = V_x X + \frac{1}{2} \sigma^2 V_{xx} X^2 + V_t$$

and write it's a corresponding terminal condition for T.

Guess and verfiy a solution of the form:

$$V(X,t) = z(t)U(X_t).$$

Then, pin down an ODE for $z(t)$. Solve the ODE and verify that the direct and indirect approaches coincide.

(c) **Modification:** Explain in words, how would you solve the problem if I give you the opposite case: you get utility $Q(X_t)$ from $t \in [0,T)$ and the process ends at $T$.

(d) Can we add the solutions from (b) and (c) to solve a problem with both, flow utility and terminal values?

**Excercise 3 [HJB Equation - Time to Leave a Set].** Assume that $X_t$ follows an O-H process. Assume that . Define a set $[A,B]$.

Compute the expected time to live the set. Assume you start at the interior of the set. Compute the expected time to touch $\partial[A,B]$.

(a) **Direct Approach:** Don't solve this, but do a sketch of how you could solve this question.

(b) **Indirect Approach:**

Solve the ODE:

$$0 = 1 + V_x X + \frac{1}{2}\sigma^2 V_{xx} X^2$$

subject to: $V(A) = V(B) = 0$.

Then, explain why:

$$V(X) = E[\tau_x | X_t = X]$$

where

$$\tau_x = \min\left\{\inf_t \{X_t = A\}, \inf_t \{X_t = B\}\right\}.$$

**Excercise 4 [Merton's Problem].** Write down a consumption savings problem in continuous time. Assume that the individual can invest in stocks or bonds. Stocks follow a Geometric Brownian motion. Bonds pay no interest rate.

1. Write the SDE for Wealth.
2. Assume CRRA. Guess and verify that the value function is of the form

   $$V(W) = AU(W).$$

   Solve for the optimal portfolio and consumption rules.
3. Assume CARA utility. Solve the model again.