Lecture 7: Continuous Time Search

1 Why Search and Matching

Many assets can be illiquid, not only do to financial frictions, but sometimes, because the environment doesn't allow to materialize gains from trade. A natural reason is that there is a buyer out there, willing to trade with a seller, but they can't find each other. A natural place to think about this frictions is the housing market. This theory is naturally employed in many billateral relations such as labor markets and the marriage market. However, a market as active as the FED funds market, operates "over-the-counter." There is no centralized sytem that clears prices and quantities simultaneously, as for example, with stocks.

We will cover a couple of topics in this lecture. First, I will begin with a review of Poisson processes. Then derive some basica Bellman equations. Then we will study two applications. The first is the seminal paper by Duffie Garleanu and Pedersen. This is a seminal paper that studies search in (Non-Walrasian) in over-the-counter markets. The theoretical contribution is to present an explicit computation of allocations and prices in a setup with dynamic search and bargaining. The model is used to account for a liquidity premium and volumes of trade. The paper is a benchmark in several recent papers, including work by Rob Shimer, Pierre Olivier Weill and Semih Uslu.

1.1 Reminder Poisson Processes

From Bernoulli to Poisson. Divide the real line into a sequence of time $t = \{1, 2, 3, ..., \infty\}$. Assume that every periods, there is a chance of an occurrence of a given event. We let the random variable $N_y = 1$ when the event has occurred by time t. This function is a random variable because it is a function defined in an underlying probability space. We let that probability be, p. Because the process is Bernoulli, it is independent by definition. Thus, the probability that an event has not occured by period t is given by:

$$\Pr\{N_t = 0\} = (1-p)^t$$

The chance that the event occurs exactly at time t+1 is:

$$\Pr\{N_{t+1} = 1 | N_t = 0\} = (1-p)^t p.$$

Observe that $\Pr\{N_{t+s} = 0 | N_t = 0\} = (1-p)^t (1-p)^{t+s-t} = (1-p)^{t+s} = \Pr\{N_{t+s} = 0 | N_t = 0\}$.

Thus,

$$\frac{\Pr\{N_{t+s}=0|N_t=0\}}{\Pr\{N_t=0\}}=(1-p)^s=\Pr\{N_s=0\}.$$

Thus, the process is stationary.

Now, we want to study what happens as we divided the real line into a finer set of points. Thus, as we set $t=\{\Delta,2\Delta,...\infty\}$. However, we study this problem the probability in a particular way. We cannot simply divide the real line into finer and finer points without adjusting the probability. Otherwise, as we continue doing so we would eventully have infinite Bernoulli trials in a single period. Instead, we are more careful and specify a ratio for the probabilities: we fix a constant

$$\lambda\Delta=p(\Delta)$$

Thus, as $\Delta\to0$, the probability shrinks, but the rate at which it does is constant. The date can be divided into the number of intervals via:

$$t=\Delta n(\Delta,t).$$

Let's now compute the chances of seeing the event at a particular date:

$$\Pr\{N_t=1|N_{t-\Delta}=0\}=(1-p(\Delta))^{n(\Delta,t)-\Delta}p(\Delta).$$

Thus, we write this expression as:

$$(1-\lambda\Delta)^{n(\Delta,t)-\Delta}\lambda\Delta.$$

Since as $\Delta\to0$, the probability of the event becomes a density, we can define the following density:

$$f(t;\lambda)=\lim_{\Delta\to\infty}\frac{(1-\lambda\Delta)^{n(\Delta,t)-\Delta}\lambda}{\Delta}\Delta.$$

I'm skipping many steps from measure theory that prove the continuity of probabilities. For now, we can compute this object:

$$\begin{aligned}&\lim_{\Delta\to\infty}(1-\lambda\Delta)^{t/\Delta}(1-\lambda\Delta)^{-\Delta}\lambda\\=\;&\left(\lim_{\Delta\to\infty}(1-\lambda\Delta)^{1/\Delta}\right)^t\lim_{\Delta\to\infty}(1-\lambda\Delta)^{-\Delta}\end{aligned}$$

and by definition of e, we obtain:

$$
\begin{aligned}
&= \left(\lim_{\Delta\to\infty}(1-\lambda\Delta)^{1/\Delta}\right)^t \lim_{\Delta\to\infty}(1-\lambda\Delta)^{-\Delta}\lambda \\
&= \lambda\exp(-\lambda t).
\end{aligned}
$$

Let's integrate this density from zero to infinity:

$$\int_0^\infty \lambda\exp(-\lambda t)\,dt = -\exp(-\lambda t)\,|_0^\infty = -0-(-1) = 1.$$

This is the density of the probability that the event occurs at time t. Naturally, the density of the event not occurrig by t can be obtained in a similar way.

Connection with Poisson Distribution. Poisson processes are understood as random $N(\omega)$ variable taking values in Z_+. These are the only processes satisfying the conditions of independent increments, stationarity and that expected number of arrivals in a given fraction of time is constant. The distribution of a poisson counting process is:

$$P(N=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

A poisson processes is defined over the real line and there distribution, namely the distribution of an at a point t of time is. That is, it is exaclty like the poisson counting process, except that we affect the probability of the counting process by where we are on the real line. Thus:

$$P(N_t=k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Additional useful property of the poisson process is its additivity charachteristic; namely that X+Y poisson with rates λ_x and λ_y, it holds that :

$$N_t^1 + N_t^2\tilde{}\text{Poisson}(\lambda_x+\lambda_y)$$

The stopping time $\tau = \inf\{s : N_{t+s} - N_t = 1\}$ has and exponential density with parameter λ, —the same as the parameter of the Poisson process that generated the stopping time:

$$\tau\tilde{}\lambda\cdot\exp(-\lambda t)$$

so obviously the stopping time $\tau = \min(\tau_1, \tau_2)$ for two independent poisson processes with rates λ_1 and λ_2 is associated with the $X_1 + X_2$ poisson random varible. Therefore, its distribution is

$$\tau \tilde{} \, (\lambda_1 + \lambda_2) \cdot \exp(-(\lambda_1 + \lambda_2) t)$$

Finally, $P[\tau = \tau_1]$ is computed as follows:

$$P(\tau_1 = t, N_t^2 = 0) = P(\tau_1 = t) \cdot P(N_t^2 = 0) = (\lambda_1 \cdot \exp(-\lambda_1 t)) \cdot \exp(-\lambda_2 t) .$$

which is the probability that N_t^1 jumps at t while N_t^2 remains at 0. Integrating over the whole domain yields:

$$\begin{aligned} \int_o^\infty (\lambda_1 \cdot \exp(-\lambda_1 t)) \cdot \exp(-\lambda_2 t)\, dt &= \int_o^\infty \exp(-\lambda_2 t)\, dt - \int_o^\infty \exp(-(\lambda_1 + \lambda_2) t) \lambda_1 dt \\ &= \frac{\lambda_1}{(\lambda_1 + \lambda_2)} \end{aligned}$$

and to check that this is correct we look at $P[\tau = \tau_1] + P[\tau = \tau_2]$, by symmetry:

$$\frac{\lambda_1}{(\lambda_1 + \lambda_2)} + \frac{\lambda_2}{(\lambda_1 + \lambda_2)} = 1.$$

2 The Hamilton-Jacobi-Bellman Equation

We now move on to study the continuous time version of Bellman equations. Let's begin first computing a net present value formula, without the maximization arguments. We try to compute:

$$V_t = \sum_{s=0}^{\infty} \beta^{t+s} U(c_{t+s})$$

where $u(c_t)$ is some constant. Then, we may a following change of variables and let $\beta = \exp(-\rho)$. Thus, ρ is approximately $1 - \beta$. Thus, we can write the expression as:

$$V_t = \sum_{t=0}^{\infty} \exp(-\rho(t+s)) U(c_{t+s}) .$$

However, we want to make the same computations as time invervales become smaller and smaller. Naturally, we need to weight those points by Δ. Thus, the value is:

$$V\left(t,\Delta\right)=\sum_{s=0:\Delta:\infty}\exp\left(-\rho\left(t+s\right)\right)U\left(c_{t+s}\right)\Delta.$$

Taking

$$\lim_{\Delta\rightarrow\infty}V\left(t,\Delta\right)=\int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds.$$

From now on, abandon the notation that Δ for the time interval and we use it to compute derivatives.

By definition of integral. Now, we want to compute how the value changes as a function of calendar time. Thus, we compute:

$$\frac{V\left(t+\Delta\right)-V\left(t\right)}{\Delta}.$$

Why this object? Because it will be incredibly useful to characterize the problems. By Definition thus, we have:

$$\begin{aligned}V\left(t+\Delta\right)-V\left(t\right) &= \int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+\Delta+s}\right)ds-\int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds\\ &= \int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+\Delta+s}\right)ds-\int_{0}^{\Delta}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds-\int_{\Delta}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds\\ &= \int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+\Delta+s}\right)ds-\int_{0}^{\Delta}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds-\int_{0}^{\infty}\exp\left(-\rho\left(z+\Delta\right)\right)U\left(c_{t+z}\right)dz\end{aligned}$$

where the last line is a change of variables $z=s+\Delta$. Thus, we have:

$$V\left(t+\Delta\right)-V\left(t\right)=\left(1-\exp\left(-\rho\Delta\right)\right)\int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+\Delta+s}\right)ds-\int_{0}^{\Delta}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds.$$

Then, we obtain:

$$\lim_{\Delta\rightarrow\infty}\frac{V\left(t+\Delta\right)-V\left(t\right)}{\Delta}=\lim_{\Delta\rightarrow\infty}\frac{\left(1-\exp\left(-\rho\Delta\right)\right)}{\Delta}\int_{0}^{\infty}\exp\left(-\rho s\right)U\left(c_{t+\Delta+s}\right)ds-\lim_{\Delta\rightarrow\infty}\frac{\int_{0}^{\Delta}\exp\left(-\rho s\right)U\left(c_{t+s}\right)ds}{\Delta}.$$

The next line just follows a number of definitions. Thus, we have:

$$\dot{V} = \lim_{\Delta \to \infty} \frac{(1 - \exp(-\rho\Delta))}{\Delta} \cdot \lim_{\Delta \to \infty} \int_0^\infty \exp(-\rho s) U(c_{t+\Delta+s}) \, ds - \lim_{\Delta \to \infty} \frac{\int_0^\Delta \exp(-\rho s) U(c_{t+s}) \, ds}{\Delta}.$$

The last term can be evaluated via Leibnitz's rule. First term, is the product of two limits. Thus, we have:

$$\begin{aligned} \dot{V} &= \lim_{\Delta \to \infty} - \underbrace{\frac{(\exp(-\rho\Delta) - \exp(-\rho \cdot 0))}{\Delta}}_{\frac{\partial \exp(-\rho t)}{\partial t}|_{t=0}} \cdot V(t) - \exp(-\rho s) U(c_{t+s})|_{s=0} \to \\ \rho V &= \dot{V} + U(c_{t+s}). \end{aligned}$$

If c is a constant, then V_t is a constant, and we have that $\dot{V} = 0$. In this case, we obtain:

$$\rho V = U(c).$$

This is nothing but the famous net present value function.

Question. What if c_t varies with time? Does this ODE have a solution?

Question. Derive the corresponding formula when interest rates are time varying.

Another Example. Assume now that C is a function of some variable S. Thus, we have $c(S)$. In addition, assume that S_t is some known function of time. In particular, we assume that

$$\dot{S} = \mu(S) S.$$

Then, we can defint the value function as $V(S,t)$, and do the same calculations.

$$\begin{aligned} \lim_{\Delta \to \infty} \frac{V(S_{t+\Delta}, t+\Delta) - V(S_t, t)}{\Delta} &= \lim_{\Delta \to \infty} \frac{(1 - \exp(-\rho\Delta))}{\Delta} \int_0^\infty \exp(-\rho s) U(c_{t+\Delta+s}) \, ds - \lim_{\Delta \to \infty} \frac{\int_0^\Delta \exp(-\rho s) U(c_{t+s}) \, ds}{\Delta} \\ &= \lim_{\Delta \to 0} \frac{(1 - \exp(-\rho\Delta))}{\Delta} \cdot \lim_{\Delta \to 0} \int_0^\infty \exp(-\rho s) U(c(S_{t+\Delta+s})) \, ds - \lim_{\Delta \to 0} \frac{\int_0^\Delta \exp(-\rho s) U(c}{\Delta} \end{aligned}$$

Now, observe that:

$$\lim_{\Delta \to \infty} \frac{V(S_{t+\Delta}, t+\Delta) - V(S_{t+\Delta}, t) + V(S_{t+\Delta}, t) - V(S, t)}{\Delta} = \dot{V} + V_s \dot{S}.$$

Thus, we obtain the following formula:

$$\rho V = U\left(c\left(S\right)\right) + V_s \mu\left(S\right)\dot{S} + \dot{V}.$$

And gain, since S_t is the only relationship to V we don't need the time formula because it become redundant. Thus, we have:

$$\rho V = U\left(c\left(S\right)\right) + V_s \mu\left(S\right)\dot{S}.$$

With a constant of Maximization. Now let's consider the following problem:

3 Duffie Garleanu Pedersen (2005, Econometrica)

3.1 Main Features of the Model

- Set of investors with 1 risk free bank account. Agent's have bounded wealth W_t
- Over the counter market for a asset consol paying a unit of consumption per year.
- A fraction s was originally endowed with the bond.
- Two types of investors:
  - l-type: δ cost in units of consumption of holding the asset.
  - h-type: no cost.
- Investors are switch types at constant rates: λ_u, intensity of switches from low to high types and λ_d the analog.
- Thus the state-space is $\tau = \{ho, hn, lo, \ln\}$.

3.2 Possible Interpretations

- Low Liquidity.
- High Financing Costs.
- Hedging reasons to sell the Asset.
- Relative Tax advantages.
- Lower Personal use of the Asset (Housing for example)

3.3 Demographic Accounting

Because at all times the populations is constant:

$$\mu_{ho}(t) + \mu_{hn}(t) + \mu_{lo}(t) + \mu_{ln}(t) = 1$$

and since the amount of assets are also constant:

$$\mu_{ho}(t) + \mu_{lo}(t) = s.$$

The last condition is very important to determine if the market stays long or short.

3.4 Meetings

- Two investors meet at rate λ, uniform across investors.
- Investors bargain sequentially as find each other.
- Investors meet market makers with intensity ρ which can be interpreted as the sum of intensity investors.
- Market makers have a particular ease to meet investors.
- Over the Counter Markets with no intermediaries are treated as $\rho = 0$.
- Hence, the measurability constraints satisfy:

$$\dot{\mu}_{lo}(t) = -(2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t),$$

$$\mu_m(t) = min\{\mu_{lo}(t), \mu_{hn}(t)\}$$

3.5 An image is worth...

The following picture summerizes the Meeting and Demographic structure:

3.6 Market Makers

- In equilibrium, low types want to sell and high types want to buy.
- When agents meet, they bargain over the prices. Similarly, investors meet marketmakers and bargain.

Figure 1:

- Investors bargaining power depends on outside options.
- Automatic Resolution of Bargaining Problem.

3.7 Demographic Accounting

We guess that there is trade from low-type with held assets to high types with no assets.

$$\dot{\mu}_{lo}(t) = -(2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t),$$

$$\dot{\mu}_{hn}(t) = -(2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) + \lambda_u u\mu_{ln}(t) - \lambda_d\mu_{hn}(t)$$

$$\dot{\mu}_{ho}(t) = (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \rho\mu_m(t)) + \lambda_u\mu_{lo}(t) - \lambda_d\mu_{ho}(t)$$

$$\dot{\mu}_{ln}(t) = (2\lambda\mu_{hn}(t)\mu_{lo}(t) + \mu_m(t)) - \lambda_u\mu_{ln}(t) + \lambda_d\mu_{hn}(t).$$

The elements in each equation refer to trading and intrinsic change of types.

4 Equilibrium

4.1 Main Result

Proposition 1 (Proposition 1)**.** *There exists a unique constant steady-state solution to (1)–(6). From any initial condition* $\mu(0) \in [0,1]^4$ *satisfying (1) and (2), the unique solution* $\mu(t)$ *to (3)–(6) converges to the steady state as* $t \to \infty$*.*

Proof. Sketch of the Proof. □

4.2 Formally The Agent's Problem

We now look at conditions that induce trade. The previous section assumed that there is trade

An agent's problem consists on:

$$U(W_t, \sigma(t), t) = \sup_{C,\theta}, E_t \int_0^\infty e^{-rs} dC_{t+s}$$

s.t.

$$dW_t = rW_t dt - dC_t + \theta_t \left(1 - \delta I_{(\sigma^\theta(t)=lo)}\right) dt - \hat{P}_t d\theta_t$$

this problem can be written in recursive form and will satisfy:

$$V(t)(t) = \sup_\theta E_t \left[\int_t^\infty e^{-r(s-t)} \left(1 - \delta I_{(\sigma^\theta(s)=lo)}\right) ds - e^{-r(s-t)} \hat{P}_\theta d\theta\right]$$

where $\hat{P}_\theta \in \{P_t, A_t, B_t\}$, and is the price that a particular agent can have access to.

4.3 In the Value Functions

To compute the time derivative of the value functions we use our digression on the poisson processes. For example, let's concentrate on the second of these equations:

$$\begin{aligned} V_{ln} &= E_t\left[e^{-r(\tau_i - t)}\left(V_{hn}\right)\right] \\ &= \int_t^\infty e^{-r(\tau_i - t)}\left(V_{hn}\right) \lambda_d \exp\left(-\lambda_d(\tau - t)\right) d\tau \\ &= \lambda_d V_{hn} \int_t^\infty e^{-(r+\lambda_d)(\tau_i - t)} d\tau \end{aligned}$$

so deriving with respect to time:

$$
\begin{aligned}
\dot{V}_{ln} &= -\lambda_d V_{hn} + (r + \lambda_d) V_{ln} \\
&= rV_{ln} - \lambda_d (V_{hn} - V_{ln})
\end{aligned}
$$

The third equation is:

$$
\begin{aligned}
V_{ho} &= E_t \left[\int_t^{\tau_l} e^{-r(s-t)} ds + e^{-r(\tau_i - t)} (V_{lo}) \right] \\
&= E_t \left[-\frac{1}{r} e^{-r(\tau_l - t)} + \frac{1}{r} e^{-r(t-t)} + e^{-r(\tau_i - t)} (V_{lo}) \right] \\
&= E_t \left[\frac{1}{r} e^{-r(t-t)} + \left(V_{lo} - \frac{1}{r} \right) e^{-r(\tau_l - t)} \right] \\
&= \int_t^{\infty} \left[\frac{1}{r} e^{-r(t-t)} - \frac{1}{r} e^{-r(\tau_l - t)} + (V_{lo}) e^{-r(\tau_l - t)} \right] \lambda_d e^{-\lambda_d (\tau_l - t)} d\tau_l
\end{aligned}
$$

so deriving with respect to time, by direct analogy of the second equation yields:

$$
\dot{V}_{ho} = rV_{ho} - \lambda_d (V_{lo} - V_{ho}) - 1
$$

The tricky part in this step is to obtain the -1 term in the LHS. This comes from the fact that the term $\frac{1}{r} e^{-r(t-t)}$ has to be derived for each t. Because the second one yields the rV_{ho} term, the firs one yields the -1 term.

The last value function:

$$
V_{hn} = E_t \left[e^{-r(\tau_l - t)} V_{\ln} I (\tau_l = \tau) + e^{-r(\tau_i - t)} (V_{ho} - P) I (\tau_i = \tau) + e^{-r(\tau_m - t)} (V_{ho} - B) I (\tau_m = \tau) \right]
$$

may be broken apart by using the indicator functions as event probabilities. We've seen in the Poisson Processes digression that the probabilities of this events are defined by the ratio of the poisson rates. Also, the events that a stopping time occurs before another are independent of time. This property is a result of the independence of increments in the original poisson process which implies the independence of the occurrence of the stopping time. Thus, that a stopping time occurs before another is independent of the point were we start the process. Hence, the problem may be written as:

$$
\begin{aligned}
= \quad & E_t\left[e^{-r(\tau_l-t)}V_{\ln}|\tau_l=\tau]\frac{\lambda_d}{\lambda_d+2\lambda\mu_{hn}+\rho}\right]+ \\
& E_t\left[e^{-r(\tau_i-t)}\left(V_{ho}-P\right)|\tau_i=\tau\right]\frac{2\lambda\mu_{hn}}{\lambda_d+2\lambda\mu_{hn}+\rho} \\
& +E_t\left[e^{-r(\tau_m-t)}\left(V_{ho}-B\right)|\tau_m=\tau\right]\frac{\rho}{\lambda_d+2\lambda\mu_{hn}+\rho}
\end{aligned}
$$

Again deriving the terms inside the expectation with respect to time yields:

$$
rV_{hn}+\left(\lambda_d+2\lambda\mu_{hn}+\rho\right)V_{hn}
$$

because the poisson rate of τ is $\left(\lambda_d+2\lambda\mu_{hn}+\rho\right)$. The conditional rate of $\tau_l|\tau_l=\tau$ has to be that of τ, so because this density is

$$
\left(\lambda_d+2\lambda\mu_{hn}+\rho\right)e^{-(\lambda_d+2\lambda\mu_{hn}+\rho)(\tau-t)}
$$

(see above digression on Poisson rates), then by direct application of Leibnitz rule:

$$
\dot{V}_{hn}=rV_{hn}-\lambda_d\left(V_{\ln}-V_{hn}\right)-2\lambda\mu_{hn}\left(\left(V_{ho}-P\right)-V_{hn}\right)-\rho V_{hn}\left(\left(V_{ho}-B\right)-V_{hn}\right)
$$

Now we turn to the first of the value Functions:

$$
V_{lo}=E_t\left[\int_t^\tau e^{-r(s-t)}\left(1-\delta\right)ds+e^{-r(\tau_l-t)}V_{ho}I\left(\tau_l=\tau\right)+e^{-r(\tau_i-t)}\left(V_{\ln}+P\right)I\left(\tau_i=\tau\right)+e^{-r(\tau_m-t)}\left(V_{\ln}+B\right)I\left(\tau_m=\tau\right)\right]
$$

we can simply use an analog argument for every term here as we followed in the previous equation to obtain the derivative w.r.t time of this equation, the result is:

$$
V_{lo}=rV_{lo}-\lambda_u\left(V_{ho}-V_{lo}\right)-2\lambda\mu_{hn}\left(V_{\ln}+P-V_{lo}\right)-\rho\left(V_{\ln}+B-V_{lo}\right)-\left(1-\delta\right)
$$

4.4 More Intuitively

We can re-express the agent's problem as a set of Bellman equations: Value functions can be written in a more natural way:

$$
V_{lo}=E_t\left[\int_t^\tau e^{-r(s-t)}\left(1-\delta\right)ds+e^{-r(\tau_l-t)}V_{ho}I\left(\tau_l=\tau\right)+e^{-r(\tau_i-t)}\left(V_{\ln}+P\right)I\left(\tau_i=\tau\right)+e^{-r(\tau_m-t)}\left(V_{\ln}+B\right)I\left(\tau_m=\tau\right)\right]
$$

$$V_{ln} = E_t \left[e^{-r(\tau_i - t)} \left(V_{hn} \right) \right]$$

$$V_{ho} = E_t \left[\int_t^{\tau_l} e^{-r(s-t)} ds + e^{-r(\tau_i - t)} \left(V_{lo} \right) \right]$$

$$V_{hn} = E_t \left[e^{-r(\tau_l - t)} V_{\ln} I \left(\tau_l = \tau \right) + e^{-r(\tau_i - t)} \left(V_{ho} - P \right) I \left(\tau_i = \tau \right) + e^{-r(\tau_m - t)} \left(V_{ho} - B \right) I \left(\tau_m = \tau \right) \right]$$

where $\tau = \min \left\{ \tau_l, \tau_m, \tau_i \right\}$ .And τ_l, τ_i, τ_m represent type-change and market stopping times respectively.

4.5 Characterization of Value Functions

The full characterization of the value functions is given by the following linear differential equations:

$$\dot{V}_{lo} = rV_{lo} - \lambda_\mu (V_{ho} - V_{lo}) - 2\mu_{hn}(P + V_{ln} - V_{lo}) - \lambda_d (B + V_{ln} - V_{lo})$$

$$\dot{V}_{ln} = rV_{ln} - \lambda_u (V_{hn} - V_{ln})$$

$$\dot{V}_{ho} = rV_{ho} - \lambda_d (V_{lo} - V_{ho}) - 1$$

$$\dot{V}_{hn} = rV_{hn} - \lambda_d (V_{ln} - V_{hn}) - 2\mu_{ho}(V_{ho} - V_{hn} - P) - \rho (V_{ho} - V_{hn} - A)$$

4.6 Reminder - Nash Bargaining

- Pie's are shared according to a parameter that measures "bargaining power".

4.7 Equilibrium Concept

Definition 2 (Equilibrium)**.** *An equilibrium is defined as a set of price paths* $\left\{ P \left(t \right) \right), A \left(t \right) \right), B \left(t \right) \right) \}$*, and mass paths* $\left\{ \mu_{lo}, \mu_{ho}, \mu_{\ln}, \mu_{hn} \right\}$ *such that:*

1. *Given Prices Paths, the Value functions satisfy the Pricing via Nash Bargaining.*

2. *Given Price paths, the mass paths, satisfy the Demographic restrictions.*

5 Characterization of Equilibrium Prices

Nash Bargaining will result in the following price system:

$$P = (V_{lo} - V_{ln})(1 - q) + (V_{ho} - V_{hn})q.$$

$$A = (V_{ho} - V_{hn})z + M(1 - z)$$

$$B = (V_{lo} - V_{ln})z + M(1 - z).$$

$$\mu_{hn} + (s - \mu_{lo}) = \lambda_\mu + u$$

6 Characterization of the Steady State

Condition 1 $s < \lambda_u/(\lambda_u + \lambda_d)$

Theorem 3 (Theorem 2)**.** *For any given initial mass distribution $\mu(0)$, there exists an equilibrium. There is a unique steady-state equilibrium. Under Condition 1,the ask, bid, and inter-investor prices are*

$$A = \frac{1}{r} - \frac{\delta}{r} \frac{\lambda_d + 2\mu_{lo}(1-q)}{r + \lambda_d + 2\lambda\mu_{lo}(1-q) + \lambda_u + 2\lambda\mu_{hn}q + \rho(1-z)}$$

$$B = \frac{1}{r} - \frac{\delta}{r} \frac{zr + \lambda_d + 2\mu_{lo}(1-q)}{r + \lambda_d + 2\lambda\mu_{lo}(1-q) + \lambda_u + 2\lambda\mu_{hn}q + \rho(1-z)}$$

$$P = \frac{1}{r} - \frac{\delta}{r} \frac{(1-q)\,r + \lambda_d + 2\mu_{lo}(1-q)}{r + \lambda_d + 2\lambda\mu_{lo}(1-q) + \lambda_u + 2\lambda\mu_{hn}q + \rho(1-z)}$$

$s < \lambda_u/(\lambda_u + \lambda_d)$. In this case,

Proof. Sketch of Proof ☐

6.1 Simulation