Sticky Inflation: Some Unpleasant New Keynesian Arithmetic

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Preliminary - Do not Circulate

Abstract

We append the expectation of a monetary-fiscal reform into a standard new-Keynesian model. Under the reform, monetary policy is temporarily obliged to provoke inflation to aid the government make its debt sustainable. After the reform, debt and inflation are stabilized again. We study the fight against inflation prior to the realization of an expected reform. Temporary increases in nominal rates carry two effects: a standard deflationary effect through aggregate demand and an inflationary effect through expected future inflation. Expected future inflation follows because higher rates increase the fiscal debt burden, signaling that greater inflation may occur in the future. While the standard demand effect can reduce inflation on impact, inflation returns more strongly through the effect on expectations (sticky inflation). An optimal commitment policy allows for trend in inflation to reduce the debt burden until a fiscal reform takes place.

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1. Introduction

Inflation became globally endemic in the aftermath of the Covid-19 pandemic. Most central banks responded to calls to raise policy rates to combat inflation. The vision behind these policy actions is that without a proper response, expectations could become unanchored, further fueling inflation. This view takes for granted that interest rate increases, by contracting aggregate demand, lower current inflationary pressures and, furthermore, signal the central bank's commitment to combat inflation. This conventional view, however, abstracts away from the fact that, with higher interest rates, national debts accumulate faster and potentially reach points where financing them solely through taxes may become unfeasible (for example Zhengyang, Lustig, Nieuwerburgh and Xiaolan, 2022). This observation is particularly disturbing in the post-pandemic years as inflation surged while most national debts rose to levels not seen in decades. The concern is that, by provoking a greater debt burden, the increase in rates may backfire through agents' expectations if they signal that a future increase in inflation will be necessary to render debts sustainable.

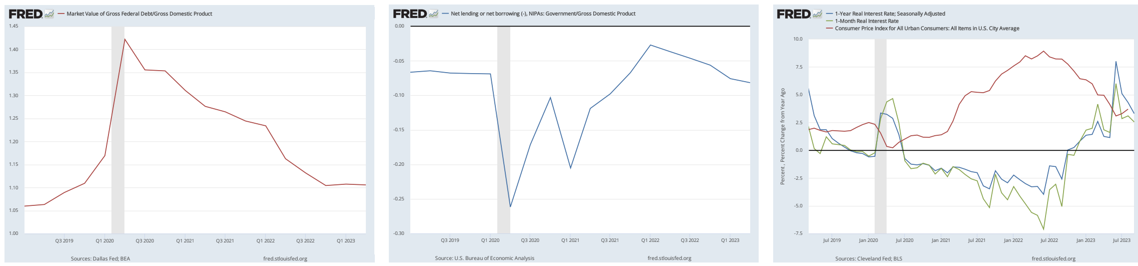
This paper studies the dynamics of inflation when nominal policy rates have the dual effect of impacting consumer demand and debt sustainability. Our goal is to analyze, in the simplest way possible, how the expectation of a greater need to inflate away government debt in the future, impacts the effects of interest-rate increases today. To that end, we study a paper-and-pencil new-Keynesian model where there is an expectation that the fiscal authority may need a monetary-fiscal reform to render its debt sustainable. During the reform, the monetary authority allows for a burst in inflation, provoking negative real rates for a fixed amount of time. After that time, debt is made sustainable and a Taylor rule also ensures the stability of the output gap and inflation thereafter. The focus of this paper is on the dynamics of inflation *prior* to such fiscal reform, as agents expect that monetary policy will provide a temporary support during the fiscal reform. The key tension is that, prior to the reform, increases in real interest rates carry two effects with opposing effects on inflation: the first is the conventional decrease aggregate demand, the second is the increase in inflation expectations that respond to the greater debt burden.

The paper makes three theoretical contributions. First, we describe the phenomenon of *sticky* *inflation.* We refer to sticky inflation as the feature that attempts to curtail inflation with interest hikes, while successful for a period of time, ends provoking higher inflation in the medium term. This occurs even though long-term inflation expectations are anchored. The reason is that any initial attempt to raise real rates provokes an increase in inflation expectations that eventually offsets the initial impact in the output gap. The second contribution is to derive optimal policies under sticky inflation. We show that these are characterized by allowing the inflation rate to have a trend. Finally, we describe how sticky inflation alters the optimal response to shocks in the new-Keynesian model.

For illustration purposes, we characterize sticky inflation in the context of shocks to a Taylor interest-rate rule that aims to close the output gap. We show how temporary increases in nominal rates can reduce inflation on impact, but eventually inflation catches up. We also show that a conventional Taylor rule leads to a constant positive inflation rate, whereas the output gap and the path of debt grow exponentially. Likewise, a policy that aims to stabilize debt, leads to an exponential growth in inflation and the output gap. All in all, attempting to stabilize one outcome variable (inflation, the output gap, or debt) produces a trend in the other variables.

In the context of sticky inflation, it remains unclear what the optimal monetary policy should be. For that reason, we investigate optimal monetary policy with commitment. We show that prior to a reform, a monetary authority interested only in minimizing the square of inflation and the output gap, should indirectly consider the squared deviation of debt relative to an inflation-neutral benchmark, into its objective function. Prior to the reform, the objective function consists of a trade-off between trends in all variables. In fact, the optimal policy allows for a trend in inflation, the output gap, and debt. We show that the optimal choice of trend is summarized by the choice of a constant real interest target prior to the reform, which has an analytical expression. Finally, we show that sticky inflation breaks the divine coincidence between the stabilization of inflation and the output gap because the inflation expectations incorporate the possibility of a monetary-fiscal reform.

The key policy message from the paper is that unless fiscal effects are solved, the sole expectation of inflation-financed debt impairs current monetary policy and changes standard results in new-Keynesian economics. The phenomenon we describe in the paper may become particular relevant as debt levels are perceived as unsustainable.



(a) Debt (b) Primary surplus (c) Real rates and inflation

Figure 1: Pre- and Post-COVID-19 Dynamics

In what contexts is this model relevant? In response to the COVID-19 crisis, the United States implemented an unprecedented fiscal expansion, resulting in the highest level of government debt (normalized by GDP) in the post-war era. Figure 1 panel (a) displays the US debt-to-GDP ratio (in market value) from 2019 Q1 to 2023 Q2.[1] The figure illustrates a sharp increase in government debt in 2020 Q2, followed by a rapid decline. By 2022 Q4, the debt had reached its pre-pandemic average, representing a decline of over 40 percentage points in a period of two and a half years. Importantly, this decline was not the result of fiscal surpluses. The primary deficit remained below its pre-pandemic average until 2022, as shown in Figure 1 panel (b). Instead, the decrease in government debt can be attributed to inflation being above the Federal Reserve's target and a negative real-interest rate on government debt during much of the post-pandemic period, as indicated in Figure 1 panel (c). Furthermore, medium-term inflation expectations, measured as the probability of persistently high inflation in the next 5 to 10 years, became unanchored, as demonstrated by Hilscher, Raviv and Reis (2022). During this time, many commentators argued that monetary policy was not sufficiently aggressive to contain the inflation rate. However, according to our theory, a more aggressive stance would have resulted in higher trend inflation. Given the private sector's belief of a non-trivial probability that the Federal Reserve would yield to the treasury and reduce the debt through inflation and negative rates, the optimal monetary policy response is to allow a short-term spike in the inflation rate to anchor the private sector's expectations in the medium-term.

Another relevant context is high-inflation countries such as Argentina, Brazil or Turkey. These

[1] A similar dynamics is observed when plotting the debt held by the private sector instead.

countries have appointed orthodox monetary policy makers that have attempted to raise real interest-rates to combat inflation. While these countries were originally successful at curtailing inflation, inflation has consistently returned. We contend that such policies cannot be successful at controlling inflation because fiscal expectations are not anchored.

Literature Review. Since the surge in inflation after the Covid-19 pandemic, understanding the drivers of inflation has, once again, regained prevalence in academic and policy debates. As a result, multiple papers have tried to explain inflationary dynamics from an analytical and quantitative standpoints. The profession is divided into two camps: one that links a nominal anchor to fiscal factors and one that does not.

On the analytical front, for example, Lorenzoni and Werning (2023b) focus on the dynamic interaction between wage and price inflation.[2] Blanchard and Bernanke (2023) present a semi-structural model that decomposes the drivers of inflation into labor-market tightness and energy shocks, but there is no scope to fiscal shocks. Similarly, Gagliardone and Gertler (2023) study the role of supply shocks and labor-market frictions, but their model leaves fiscal policy as unexplained demand factors. In all of these papers, the fiscal side, is implicitly detached from nominal variables, leaving no room to study the interaction between inflation and fiscal solvency. This recent stream of papers follows the new-Keynesian tradition that typically abstracts away from how debt-financing impairs monetary policy.

The interaction between monetary policy and fiscal solvency is vast and is part of core textbook material, (e.g., Ljungqvist and Sargent, 2018). The textbook approach abstracts away from nominal rigidities and assumes that deficits are financed with transfers of nominal balances of money. Leeper (1991) studies the interaction between monetary policy and fiscal solvency in economies where monetary policy follows a nominal interest-rate rule, that does not satisfy the Taylor principle. A key feature of this theory is that government debt is nominal. Together with Leeper (1991), Woodford (1998); Cochrane (1998) show that under flexible-price, the price-level will jump in response to fis-

[2]On related work, Lorenzoni and Werning (2023a) shows how inflationary spirals can emerge from strategic interactions between different parties that set wages in a staggered fashion.

cal news, thus providing a fiscal-theory of the price level (FTPL).[3] These papers also show that in such environments, increases in nominal policy rates lead to counter-factual increases in inflation, following the logic of Fisherian effects. In turn, Woodford (2001); Cochrane (2001) show that with long-term debt, increases in nominal policy rates lead to decreases in inflation, overturning the original counterfactual Fisherian effect. However, Sims (2011) that the increase in nominal rates can reduce inflation only in the short run, but eventually raises inflation.

This paper shares the emphasis on how fiscal solvency impacts inflation. However, our setting differs in important dimensions: we focus on the behavior in Phase I, where monetary policy is active and the Taylor principle is satisfied, in contrast to papers in the FTPL tradition. We find a similar a "stepping-on-a-rake" as the one in Sims (2011), but here the effect does not follow from the valuation of nominal long-term debt. Rather, the result follows from the confluence of two forces: a standard aggregate demand effect and an the effect on higher interest-rate burden.

A second generation studies the interaction between monetary policy and fiscal solvency that allows for sticky prices (e.g. Sims, 2011; Bianchi and Melosi, 2017; Leeper and Leith, 2016; Caramp and Silva, n.d.)). A common theme among these papers is that with nominal rigidities, interest-rate shocks and fiscal shocks do not lead to jumps in prices, but to persistent responses in inflation. Most of analytical work in this area makes an assumption regarding fiscal or monetary dominance although regime switching and expectations of policy changes are present in quantitative work.[4] Among these papers, the closest to ours is Bianchi, Faccini and Melosi (2023). In the setting in Bianchi et al. (2023), some fiscal shocks are adjusted with taxes and others are not, whereas monetary policy is always guided by a Taylor rule. In their setting, some fiscal shocks provoke inflation and the authors estimate that such unfunded fiscal shocks have been key drivers of inflation in the US. Our approach is similar in that fiscal shocks funded with a combination inflation and taxes during reform phases. Our contribution is to present a simple environment that allows for analytic results, as do Werning (2012) or Cochrane (2017) in the context policies at the zero-lower bound. We make two analytical contributions: we showcase the stepping-on-a-rake phenomenon and show that monetary

[3] In this papers, the timing of taxes does not matter as Ricardian equivalence holds.

[4] See for example, Davig and Leeper (AER 2007); Chung, Davig, and Leeper (JMCB 2007), Bianchi (Restud 2013), Bianchi and Melosi (NBER Annual 2013, AER 2017) (Bianchi and Melosi, 2017).

policy before a fiscal reform allows for trend inflation.

Our study of optimal policies also relates to some normative work in this area. Leeper, Leith, and Liu (JME 2021) and Leeper and Zhou (JME 2023) study optimal long-term debt policy. See also Leeper and Leith (2016). In these articles, the level and maturity of debt plays an important role, but the distinction between active and passive regimes disappear when considering optimal policy.[5] This is not the case in our paper because we focus on the dynamics prior to a fiscal-monetary reform.

A key feature of our environment is that reforms happen in the future, so the expectation component of inflation is key in determining current inflation. This is a common feature in other papers such as Carvalho, Moench and Preston (forthcoming) and Eusepi and Preston (2012). In particular, Eusepi and Preston (2012) also show that in similar environments inflation will trend. On the empirical front, a number of papers have found that long-term forecasts are responsive to monetary shocks. In the US, Nakamura and Steinsson (n.d.) show that increases in policy rates reduce long-term forecasts of inflation. While this correlation is contrary, to our theory, it demonstrates that long-term inflation expectations are endogenous to policy. There is no reason why the relation between increases in policy rates reduce long-term forecasts of inflation should remain stable if economies enter unsustainable debt levels. Coibion, Gorodnichenko and Weber (2022) study a randomized-control trial and argue that news about future debt leads households to anticipate higher inflation, both in the short run and long run, and induces households to increase their spending. In a long sample covering multiple countries, Brandao-Marques, Casiraghi, Gelos, Harrison and Kamber (2023) shows that surprise increases in debt levels raise long-term inflation expectations predominantly in emerging markets. Moreover, consistent with our theory, they find that the effects are stronger when initial debt levels are already high. While developing countries who have traditionally held much higher debt levels, it is possible for developed economies to follow that path.[6]

2. Model

[5] But typical fiscal theory ingredients may play a larger or smaller role. Contrast between Schmitt-Grohe and Uribe (2004) - inflation plays minor role with sticky prices - vs Leeper and Zhou (2023) -

[6] de Mendonca and Machado (2013) perform a similar study focusing on the case of Brazil.

2.1 Environment

We cast the model in continuous-time. The economy is populated by households, firms, and a government. The government engages in a fiscal expansion, while households and firms face uncertainty regarding the reaction of fiscal and monetary authorities. We describe next the behavior of the government, households, and firms in detail.

Government. The government is comprised of a fiscal and a monetary authority. The fiscal authority sends lump-sum transfers T_t to households, or levy taxes if $T_t < 0$, and issues short-debt debt whose real value is denoted by B_t. The monetary authority sets the nominal interest rate i_t. The government's flow budget constraint is given by

$$\dot{B}_t = (i_t - \pi_t)B_t + T_t, \tag{1}$$

given $B_0 > 0$, where π_t denotes the inflation rate.

We assume that fiscal transfers are given by

$$T_t = -\rho B_t + \Psi_t, \tag{2}$$

where ρ denotes the interest rate in the zero-inflation steady state. The economy will be in steady state if the fiscal authority sets $\Psi_t = 0$ for all $t \geq 0$.

We are interested in the effects of a fiscal expansion. In a first stage, which we refer to as Phase I, we assume that $\Psi_t > 0$ until one of two random events happen. First, with Poisson intensity $\overline{\theta}$, the fiscal authority switches to a regime where $\Psi_t = 0$. As the fiscal expansion was debt-financed, this corresponds to a fiscal adjustment where taxes end up permanently higher to finance the accumulated debt. In this case, we assume the monetary authority implements zero inflation, so the economy immediately jumps to a steady state. If this was the only possibility, then the fiscal expansion would be fully backed by future taxes, despite the timing of the adjustment being uncertain. However, we assume that, with Poisson intensity θ^*, the economy switches to a regime where the fiscal expansion is not fully backed by future taxes. In particular, the monetary authority maintains

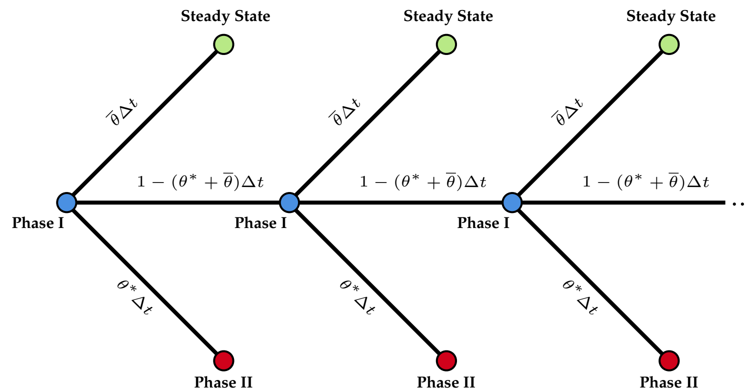


Figure 2: Timeline of events

interest rates sufficiently low for T^* periods, until debt reaches a sustainable level B_{ss}. This period of low interest rates is required to bring the debt back to a sustainable level, given the limited fiscal support. We henceforth refer to this second stage as Phase II.

Figure 2 summarizes the timeline of events. Over a small time interval Δt, a fiscal adjustment happens, and the economy goes to a steady state, with probability $\overline{\theta}\Delta t$. The economy goes to Phase II with probability $\theta^*\Delta t$, where the monetary authority provides a temporary support to the fiscal authority. Finally, the economy remains in Phase I with the remaining probability, where the fiscal authority engages in a fiscal expansion.

During Phase I, the monetary authority is free to choose its interest rate policy. In particular, we assume that nominal rates satisfy a Taylor rule:

$$i_t = \rho + \phi \pi_t + u_t, \tag{3}$$

where $\phi > 1$. The fact that the Taylor coefficient satisfies $\phi > 1$ implies the economy is in an active monetary regime, in the terms of Leeper (1991). The presence of the monetary disturbance u_t provides freedom to the monetary authority to choose how to react to the fiscal expansion. We will later consider different policy reactions by the monetary authority which implicitly correspond to different choices of the disturbance u_t. Notice that u_t captures the *response* of the monetary authority to the fiscal expansion, so we refer to it as a disturbance to the policy rule instead of a shock. We refer to the case with $u_t = 0$ as the undisturbed Taylor rule.

Notation. We index variables in Phase II using an asterisk ($*$) superscript whereas variables during Phase I do not carry that superscript. For example, π_t represents inflation at time t of Phase I whereas π_t^* is inflation at time t since the start of Phase II. Variables in the steady state are denoted by an upper bar. For example, consumption in steady state is denoted by $\overline{C}$.

Households. The problem of a household is given by

$$V_t(B_t) = \max_{[C_s, N_s]_{s \geq t}} \mathbb{E}_t^h \left[\int_t^{\tilde{t}} e^{-\rho(s-t)} \left(\log C_s - \frac{N_s^{1+\varphi}}{1+\varphi} \right) dt + e^{-\rho \tilde{t}} \tilde{V}_{\tilde{t}}(B_{\tilde{t}}) \right],$$

subject to

$$\dot{B}_t = r_t B_t + \frac{W_t}{P_t} N_t + D_t + T_t - C_t,$$

and a No-Ponzi condition $\lim_{T \to \infty} \mathbb{E}_t^h[\eta_T B_T] = 0$, where η_t denotes the stochastic discount factor (SDF) in this economy. B_t denotes the real value of bonds held by households, $r_t = i_t - \pi_t$ is the real interest rate, W_t is the nominal wage, P_t is the price level, and D_t are dividends payed by firms. The random time $\tilde{t}$ denotes the first arrival time of one of the Poisson events, either the fiscal adjustment or the switch to Phase II, whatever happens first. $\tilde{V}_t$ denotes the value function after the switch.

We do not impose rational expectations. To the extent that such fiscal expansions, and the associated changes in policy regime, correspond to potentially rare events, it may difficult for households to ascertain the true probabilities of policy changes. Therefore, we assume that, under households' subjective beliefs, the economy switches to Phase II with Poisson intensity θ_h^* and the economy moves to a steady state with Poisson intensity $\overline{\theta}_h$. The households' Euler equation during Phase I is

$$\frac{\dot{C}_t}{C_t} = (i_t - \pi_t - \rho) + \theta_h^* \left[\frac{C_t}{C_t^J} - 1 \right] + \overline{\theta}_h \left[\frac{C_t}{\overline{C}} - 1 \right], \tag{4}$$

where $\overline{C}$ denotes consumption in steady state and C_t^J denotes consumption when the economy switches to Phase II. It captures a potential jump in consumption after the change in regime. See

Appendix A for a derivation. In the absence of policy uncertainty, $\theta_h^* = \overline{\theta}_h = 0$, we obtain the standard Euler equation. The last two terms in the expression above capture the effects of the change in policy regime.

Labor supply is given by the usual intra-temporal condition:

$$\frac{W_t}{P_t} = C_t N_t^{\varphi}. \tag{5}$$

Firms. There are two types of firms in the economy: final-goods producers and intermediate-goods producers. Final goods are produced by competitive firms according to the production function $Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\epsilon}{\epsilon-1}} di\right)^{\frac{\epsilon-1}{\epsilon}}$, where $Y_{i,t}$ denotes the output of intermediate $i \in [0,1]$. The demand for intermediate good i is given by $Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon} Y_t$, where $P_{i,t}$ is the price of intermediate i, $P_t = \left(\int_0^1 P_{i,t}^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$ is the price level, and Y_t is the aggregate output.

Intermediate-goods producers have monopoly over their variety and operate the technology $Y_{i,t} = AN_{i,t}$, where $N_{i,t}$ denotes labor input. Firms are subject to quadratic adjustment costs on price changes, so the problem of intermediate i is given by

$$Q_{i,t}(P_i) = \max_{[\pi_{i,s}]_{s \geq t}} \mathbb{E}_t^f \left[\int_t^{\tilde{t}} \frac{\eta_s}{\eta_t} \left(\frac{P_{i,s}}{P_{i,t}} Y_{i,s} - \frac{W_s}{P_s} \frac{Y_{i,s}}{A} - \frac{\varphi}{2} \pi_{i,s}^2 \right) ds + \frac{\eta_{\tilde{t}}}{\eta_t} \tilde{Q}_{i,\tilde{t}}(P_{i,\tilde{t}}) \right], \tag{6}$$

subject to $Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon} Y_t$ and $\dot{P}_{i,t} = \pi_{i,t} P_{i,t}$, given $P_{i,t} = P_i$, where φ is the price adjustment cost parameter and $\tilde{t}$ denotes the random time at which there is a policy regime.

Firms' beliefs are allowed to be different from households' beliefs. This is consistent with the evidence in Candia, Coibion and Gorodnichenko (2023), who show that firms' expectation often deviates significantly from the expectation of households or professional forecasters. Therefore, we assume that firms expect the fiscal adjustment to happen with Poisson intensity $\overline{\theta}_f$ and that the economy switches to Phase II with intensity θ_f^*.

We show in Appendix A that the optimality condition for firms implies the following non-linear

New Keynesian Phillips curve (NKPC):

$$\dot{\pi}_t = (i_t - \pi_t)\,\pi_t + \theta_f^* \frac{\eta_t^J}{\eta_t}\left(\pi_t - \pi_t^J\right) + \overline{\theta}_f \frac{\overline{\eta}}{\eta_t}\pi_t - \frac{\epsilon\varphi^{-1}}{A}\left(\frac{W_t}{P_t} - (1-\epsilon^{-1})A\right)Y_t, \tag{7}$$

where π_t^J denotes the *jump inflation term*, corresponding to the inflation rate after the economy switches to Phase II, and similarly for the jump SDF term η_t^J. $\overline{\eta}$ corresponds to the SDF in steady state.

2.2 The log-linear system

We focus on a log-linear solution around the zero-inflation steady state. The steady-state economy corresponds to the case $\Psi_t = 0$ and $u_t = 0$, so $B_t = \overline{B}$, $C_t = \overline{C}$, $i_t = \rho$, and $\pi_t = 0$, where $\overline{B}$ corresponds to the initial condition for government debt and $\overline{C}$ is the steady-state level of consumption, as defined in Appendix A. During Phase I, we assume $\Psi_t = \overline{B} \times \psi_t$, for $\psi_t > 0$, so there is a fiscal expansion. The linearized government's budget constraint during Phase I is given by:

$$\dot{b}_t = i_t - \pi_t - \rho + \psi_t, \tag{8}$$

where $b_t = \frac{B_t - \overline{B}}{\overline{B}}$. If the real rate is kept at its natural level, $i_t - \pi_t = \rho$, then government debt grows over time due to the fiscal transfers ψ.

During Phase II, the fiscal shocks are set to zero, $\psi_t^* = 0$. The monetary authority implements a constant real interest rate r^* for the first T^* periods of Phase II. In particular, the monetary authority chooses $i_t^* = r^* + \pi_t^*$ such that $b_{T^*}^* = \frac{B_{ss} - \overline{B}}{\overline{B}}$. The flow budget constraint for the first T^* periods of Phase II is then given by:

$$\dot{b}_t^* = r^* - \rho, \tag{9}$$

where b_t^* denotes the level of debt after t periods the economy switched to Phase II.

The linearized Euler equation during Phase I is given by

$$\dot{x}_t = i_t - \pi_t - \rho + \theta_h x_t - \theta_h^* x_t^J, \tag{10}$$

where $x_t \equiv \frac{Y_t - \overline{Y}}{\overline{Y}}$ denotes the output gap and $\theta_h \equiv \theta_h^* + \overline{\theta}_h$. As there is no further uncertainty in Phase II, the Euler equation is simply $\dot{x}_t^* = i_t^* - \pi_t^* - \rho$.

Policy uncertainty leads to a *discounted Euler equation*, similar to other forms of uncertainty, such as the uninsurable idiosyncratic income risk of McKay, Nakamura and Steinsson (2016) or the aggregate disaster risk in Caramp and Silva (2021). To see the role of discounting, we can integrate forward the Euler equation to obtain:

$$x_t = -\int_0^\infty e^{-\theta_h s}(i_{t+s} - \pi_{t+s} - \rho)ds + \theta_h^* \int_0^\infty e^{-\theta_h s} x_{t+s}^J ds, \tag{11}$$

so the effect of changes in future interest rates gets discounted by θ_h. The second term in the expression above corresponds to an *expectation effect*.[7] It captures the impact on x_t of the expectation of a jump in the output gap when the economy switches to Phase II.

The linearized NKPC is given by

$$\dot{\pi}_t = (\rho + \theta_f)\pi_t - \kappa x_t - \theta_f^* \pi_t^J, \tag{12}$$

where $\theta_f \equiv \theta_f^* + \overline{\theta}_f$ and $\kappa > 0$ is a coefficient defined in the appendix. In contrast to the standard formulation of the NKPC, inflation dynamics depends not only on the current value of the output gap x_t, but also on firms' inflation expectation induced by the potential change in policy regime. Integrating the NKPC forward, we obtain the inflation rate

$$\pi_t = \kappa \int_t^\infty e^{-(\rho+\theta_f)(s-t)} x_s ds + \theta_f^* \int_t^\infty e^{-(\rho+\theta_f)(s-t)} \pi_s^J ds. \tag{13}$$

This expression shows the role of the output gap and of the expectation term. These expectation effects play an important in shaping the trade-offs faced by the monetary authority when responding to a fiscal expansion.

[7] See e.g. Leeper and Zha (2003) for a definition and discussion of expectation-formation effects.

3. Three policy experiments

In this section, we consider three different policy experiments where the monetary authority attempts to stabilize the output gap, inflation, or government debt. These experiments illustrate the implications of different policy responses by the central bank to a fiscal expansion that is expected not to be fully backed by future taxes.

The behavior of firms' expectations will be central in all three cases. To isolate the role of these expectations, we make the simplifying assumption that $\theta_h^* = \overline{\theta}_h = 0$, so the Euler equation behaves as in the standard New Keynesian model. Given that the implications of a discounted Euler equation have been already extensively studied in the literature, in this section we focus on the role of firms' expectations in the NKPC. We revisit the role of households' expectation effects in Section 4. Without loss of generality, we further assume that $\theta_f^* = \theta^*$ and $\overline{\theta}_f = \overline{\theta}$.

3.1 Output gap stabilization

In our first policy experiment, we assume that the monetary authority stabilizes output gap, that is, $x_t = 0$ during Phase I. We later describe the path of monetary disturbances u_t required to implement this outcome.

Phase II. During Phase II, the monetary authority maintains the real interest rate low for T^* periods and implements zero inflation after that. Debt evolves according to $b_t^* = b_0^* + (r^* - \rho)t$ for $t \geq T^*$. To ensure that debt reaches the sustainable level after T^* periods, the real interest rate must satisfy the condition:

$$r^* = \rho - \frac{b_0^* - b^n}{T^*}, \tag{14}$$

where $b^n \equiv \frac{B_{ss} - \overline{B}}{\overline{B}}$ denotes the natural level of debt, that is, the initial level of debt at Phase II such that the real rate, and ultimately output and inflation, immediately jumps to its steady-state level.

The monetary authority implements zero inflation when the sustainable debt level is achieved, that is, $\{x_{T^*}^*, \pi_{T^*}^*\} = \{0, 0\}$. Given this terminal condition, and the Euler equation $\dot{x}_t^* = r^* - \rho$, we obtain the output gap:

$$x_t^* = (r^* - \rho)(t - T^*), t \in [0, T^*], \tag{15}$$

From the NKPC and the expression for the output gap, we obtain the inflation rate:

$$\pi_t^* = \kappa(r^* - \rho) \int_t^{T^*} \exp(-\rho(s-t))(s - T^*) ds.$$

For an economy that switches to Phase II after t periods, initial debt is given by $b_0^* = b_t$. Using the expression for the real rate, we can then express initial inflation in terms of a *debt gap* $b_t - b^n$:

$$\pi^*(b_t) \equiv \kappa \Phi (b_t - b^n), \tag{16}$$

where the coefficient Φ is defined as follows:

$$\Phi \equiv \int_0^{T^*} \exp(-\rho s) \left(1 - \frac{s}{T^*}\right) ds > 0.$$

Equation (16) shows that initial inflation in Phase II depends on the amount of debt inherited from Phase I. In particular, what is relevant is $b_t - b^n$, the difference of the government to its natural level. Equivalently, $b_t - b^n$ can be interpreted as the part of government debt that is not fully backed by future taxes. We find that the larger the initial level of government debt relative to its natural level, the lower the real rate must be, and the higher is the inflation rate. The coefficient Φ captures the pass-through from debt to inflation and it captures the forward-looking component of the reform. This pass-through from debt to inflation depends on the specifics of the Phase II reform, and does not depend on monetary policy during Phase I. As long as we can cast different reforms into a single pass-through coefficient, we can always change Φ for a different constant and reach the same conclusions regarding Phase I.

Phase I. We assume that the fiscal authority sets $\psi_t = \psi$, a constant fiscal expansion while in Phase I. To stabilize the output gap during Phase I, $x_t = 0$, the real rate must satisfy $i_t - \pi_t = \rho$. Government debt is then given by $b_t = b_0 + \psi t$. The following Lemma presents a formula for the jump inflation

π_t^J term in the Phillips curve of Phase I (Equation 12):

Lemma 1 (Jump Inflation)**.** *Inflation path is increasing in the time of Phase I. In particular, the jump inflation can be written as:* $\pi_t^J = \kappa\Phi\left(b_0 + \psi t - b^n\right)$*.*

Jump inflation tracks the path of debt relative to natural. In particular, jump inflation π_t^J grows at rate $\dot{\pi}_t^J = \kappa\Phi\psi > 0$. Recall that $\kappa\Phi$ is the pass-through rate from the debt gap to inflation and that debt trends at rate ψ. Given jump inflation, and the assumption that the monetary authority stabilizes the output gap, we obtain the inflation rate using the NKPC (Equation 13):

$$\pi_t = \theta^* \int_t^\infty e^{-(\rho+\theta)(s-t)} \pi_s^J ds.$$

Using the expression for jump inflation (Equation 16), the solution to inflation is given by

$$\begin{aligned} \pi_t &= \theta^* \kappa\Phi \int_t^\infty e^{-(\rho+\theta)(s-t)} (b_t + \psi(s-t) - b^n) ds \\ &= \frac{\theta^* \kappa\Phi}{\rho+\theta} \left[b_t - b^n + \frac{\psi}{\rho+\theta} \right]. \end{aligned} \tag{17}$$

Therefore, inflation inherits the trend in government debt.

Proposition 1 (Trend in Inflation)**.** *Suppose the monetary authority eliminates the output gap. Then, inflation satisfies:*

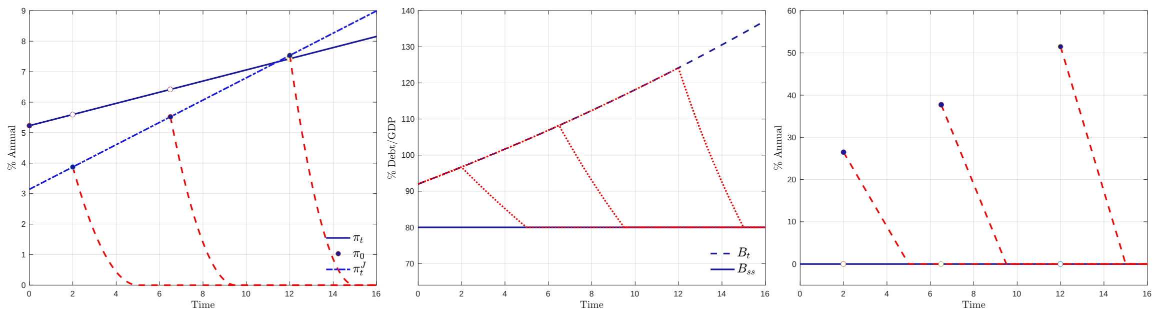
$$\pi_t = \pi_0 + \frac{\theta^* \kappa\Phi}{\rho+\theta} \psi t, \tag{18}$$

where $\pi_0 = \frac{\theta^*\kappa\Phi}{\rho+\theta}\left[b_0 - b^n + \frac{\psi}{\rho+\theta}\right]$*.*

Proposition 1 shows that, in order to stabilize the output gap, the monetary authority must accept an inflation rate that increases over time. Notice that the inflation satisfies:

$$\dot{\pi}_t = \frac{\theta^*}{\rho+\theta} \dot{\pi}_t^J.$$

That is, prior to the monetary-fiscal reform, inflation grows proportionally to the growth in jump inflation. That is, *inflation is sticky*. Because jump inflation trails with the path of debt, inflation will



(a) Inflation

(b) Debt

(c) Output Gap

Figure 3: Pre and Post Reform Equilibrium Objects

Note: Red lines correspond to reform paths that occur at different points in time.

also trail with the path of debt. This tells us something potentially deep: inflation features a trend which will be a scaled down version of the path of real debt. As a result, an attempt to combat inflation in the short run may be offset by a change in the level component of trend inflation. We formalize this intuition below.

Expression (18) also captures the importance of expectations of how the fiscal expansion will be financed. If firms believe that it is much more likely that the fiscal expansion will be financed by an increase in future taxes, $\overline{\theta} \gg \theta^*$, then the impact on inflation will be greatly attenuated (recall that $\theta = \overline{\theta} + \theta^*$). In contrast, if firms attribute a greater chance to the financing of the fiscal expansion requiring some degree of monetary accommodation, $\theta^* > \overline{\theta}$, these effects are much more pronounced.

An Example of the Dynamics. To fix ideas, Figure 3 shows the typical paths of inflation, debt, and the output gap, during Phase I and Phase II. For simplicity, we assume that $\overline{\theta} = 0$. In Panel (a), we find an example of a path of inflation which is plotted together with its corresponding jump inflation term (the dot dashed term). The red dashed curves represent different inflation levels corresponding to different dates of the arrival of Phase II. Notice that when Phase II is initiated, inflation jumps to the jump inflation term. If the reform happens early, inflation may actually jump downwards. This happens because inflation is the net-present discounted of all future jump inflation which increase over time with debt. If the reform happens early, inflation jumps downward because the initial jump inflation is lower than the net-present expected discounted value of future jump inflation.

Panel (b) shows the path of debt, together with the paths for debt corresponding to reforms that arrive at different dates. Prior to the reforms, debt grows linearly at rate ψ until a reform occurs. At each possible reform date, debt trends downwards toward its sustainable level in exactly T periods. Notice how the slope of debt after the reform is steeper, the later the reform. This reflects that, as reforms take the same amount of time, reforms that begin with higher debt levels require lower negative real interest-rate. Panel (c) shows the output gap. The later the reform, the largest the response of the output gap and, hence, a greater labor wedge. This effect results from the feature that later reforms correspond to more negative rates and higher inflation. A key lesson is that the later the reform, the greater the labor wedge.[8]

3.2 Inflation stabilization

In the previous derivation, we assumed that there are no attempts to fight inflation in Phase I, as the monetary authority is focused on stabilizing the output gap. We showed that in that case, inflation is sticky. We now investigate the effects of a temporary attempt to fight inflation while keeping the protocol in Phase II as above. In turn, during Phase I, we assume that the monetary authority attempts to run a contractionary policy to contain inflation by temporarily raising rates, such that the real rate satisfies $r_t - \rho = e^{-\theta_r t}(r_0 - \rho)$, for a given initial rate $r_0 > \rho$ and persistence parameter $\theta_r \geq 0$. We use a superscript og to denote the policy with constant real rate, corresponding to the output-gap stabilization policy derived in the previous section.

The output gap is given by

$$\dot{x}_t = r_t - \rho \Rightarrow x_t = -\frac{1}{\theta_r}(r_t - \rho), \tag{19}$$

where we used the terminal condition $\lim_{t\to\infty} x_t = 0$, a form of long-run neutrality. As $r_t > \rho$, the output gap is negative at all dates.

[8] Our model can be easily adapted to admit trend growth. Since this only affects discount rates, the observation is true even in a growing economy.

In turn, the path of debt is given by

$$b_t = b_t^{og} + \frac{1 - e^{-\theta_r t}}{\theta_r}(r_0 - \rho), \tag{20}$$

where $b_t^{og} = b_0 + \psi t$ corresponds to the debt level under the output-gap stabilization policy.

Next, we solve for inflation using the NKPC of Phase I (Equation 13):

$$\pi_t = \kappa \int_t^{\infty} e^{-(\rho+\theta)(s-t)} x_t ds + \theta^* \int_t^{\infty} e^{-(\rho+\theta)(s-t)} \pi_t^J ds.$$

The solution satisfies a superposition principle: we can super-impose a path to the output gap and one for jump inflation and we will obtain another path that satisfies the Phillips curve. We now have that inflation is given by the sum of two effects, the fight-inflation effect and a jump-inflation effect. We can write inflation, relative to the solution without a shock, as:

$$\pi_0 - \pi_t^{og} = F_t^{\pi} + J_t^{\pi}.$$

The first term reflects the effect on the output gap that results from the policy. We can label this the fight-inflation term:

$$F_t^{\pi} \quad = \quad -\frac{\kappa}{\theta_r} \frac{r_t - \rho}{\rho + \theta + \theta_r}. \tag{21}$$

This term is negative, since $r_t > \rho$, increasing over time, and converges to zero as $t \to 0$. Clearly, an increase in r_t above the natural rate ρ mitigates inflation, as in standard versions of the New Keynesian model.

The second term, the jump inflation term, is:

$$J_t^{\pi} \quad = \quad \frac{\theta^* \kappa \Phi}{\theta_r} \left[\frac{1}{\rho + \theta} - \frac{e^{-\theta_r t}}{\rho + \theta + \theta_r} \right] (r_0 - \rho). \tag{22}$$

using the fact that $\pi_t^J = \kappa \Phi (b_t - b^n)$ and $b_t - b_t^{og} = \frac{1 - e^{-\theta_r t}}{\theta_r}(r_0 - \rho)$. The jump inflation term is positive, increasing over time, and converges to a positive limit as $t \to 0$. Jump inflation is related to government debt, which is increasing in the level of the real rate. In a nutshell, debt accumulation pressures prices upwards.

Which effect dominates, the fight inflation or the jump inflation, depends on the persistence, but not the size, of the policy. The source of the opposing effects is that an increase in rates, while provoking a decline in the output gap, also provokes an increase in the real path of debt, leading to an increase in the expectation of higher inflation. Next, we show that, while the monetary authority can successfully fight inflation at time zero, inflation will always return.

We can verify from the expressions above that when $r_0 = \rho$, both F_0^π and J_0^π are zero. More generally, these expressions allow us to analyze the effects at $t = 0$, once the fight-inflation strategy is put in place. We have the following condition for a successful policy.

Proposition 2 (Successful fight condition)**.** *Suppose $r_0 > \rho$. The policy reduces inflation at time zero if and only if:*

$$\theta_r < \frac{\rho + \theta}{\theta^* \Phi}.$$

The proposition shows that a successful fight-inflation policy requires that the fight-inflation effect be greater than the jump-inflation effect. The condition above shows that the increase in interest rates must be persistent enough for the fight-inflation term to dominate at $t = 0$. Hence, it is possible to fight inflation and reduce it at time zero. How persistent the increase in real rates needs to be depends on the expectation parameters. If $\overline{\theta}$ is large, then firms expect the fiscal expansion to be likely to be resolved by higher future taxes, so even relatively transitory interest rate hikes are successful in reducing inflation. In contrast, if θ^* is large, then firms believe it is likely that some monetary support is necessary to stabilize debt, so the increase in real rates needs to be very persistent to bring down inflation at the initial period.

Next, we consider what happens to inflation as we move away from $t = 0$. In general, inflation π_t depends on the sum of the fight and jump inflation terms. Again, the two effects oppose each other. Given that the fight-inflation term converges to zero, while the jump-inflation term converges to a positive constant, we arrive at the following unpleasant "stepping-on-a-rake" result:

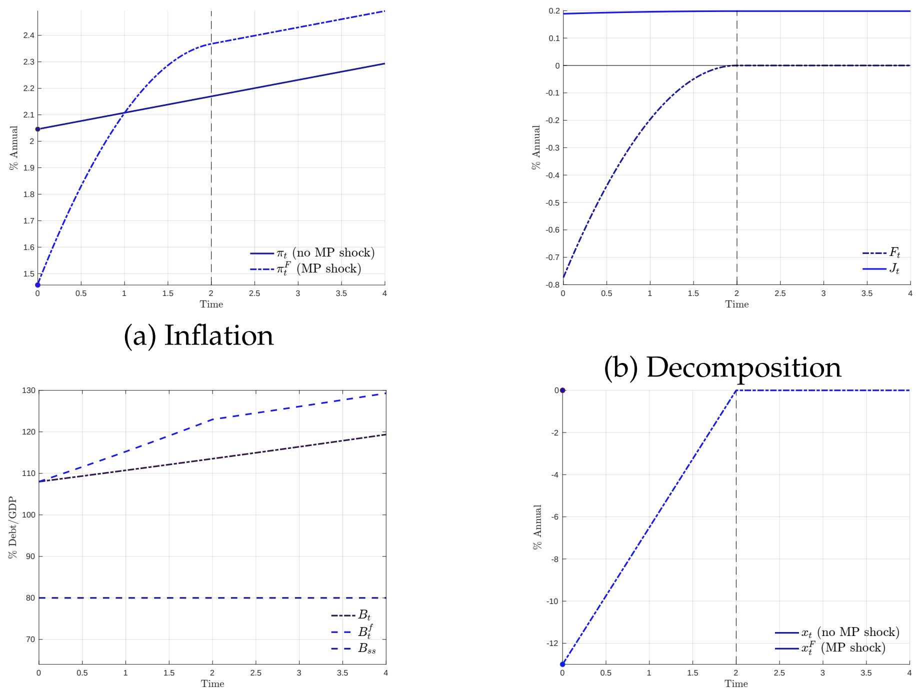
Proposition 3 (Stepping on a Rake)**.** *There always exists a $\hat{T}$ sufficiently large such that $\pi_t > \pi_t^{og}$ for*

$t > \hat{T}$. Moreover, $\hat{T} > 0$ if the condition in Proposition 2 is satisfied.

The main message of this proposition is that while a policy may be successful at curtailing inflation temporarily, eventually, inflation comes back stronger. Again, inflation is sticky! The reason is that the effect on the output gap eventually fades away whereas the effect on the path of government debt grows faster. As a result, the jump inflation term eventually dominates. There is no possibility to permanently control inflation, prior to the reform of Phase II. Sims (2011) call that boomerang feature of inflation "stepping-on-a-rake." Importantly, we find this result under very different conditions. Sims (2011) obtains the stepping-on-a-rake result in a model with long-term bonds and a Taylor coefficient $\phi < 1$, while we obtain this result under opposite conditions: short-term bonds and a Taylor coefficient $\phi > 1$. As shown by Cochrane (2018), long-term bonds are strictly necessary to obtain this result in the absence of the expectation effects we consider in this paper, which indicates that our result operates through a different mechanism than the one originally obtained by Sims (2011).

An Example of the Fight Inflation Policy. Figure 4 shows the paths of inflation, debt, and the output gap, considering an attempt to fight inflation in Phase I. We again set $\overline{\theta} = 0$. In Panel (a), we find an example of a path of inflation together with a path of inflation corresponding to a temporary increase in policy rates for 4 years. Notice how the example illustrates Propositions 2 and 3. While the strategy is successful at combating inflation earlier on, inflation returns a year into the policy. Panel (b) shows why: it plots the fight inflation and jump inflation components. The fight inflation component is initially strong but eventually dies out, as the effect on aggregate demand vanishes. The expected inflation effect is initially weak, but continues to increase as the path of log debt increases at a higher trend level. Panel (c) shows the path of debt. Debt is growing linearly without the policy. With the policy, debt picks up in response to the higher real interest rates to eventually trend at the same rate. Finally, Panel (d) shows the output gap (or labor wedge) corresponding to the policy attempt. The policy is contractionary, as dictated by the fight inflation effect.

The example shows that with an expectation of a fiscal reform, attempts to curtail inflation have undesired effects. Any initial increase in rates eventually leads to higher inflation, unlike the canon-



(a) Inflation

(b) Decomposition

(c) Debt

(d) Output Gap

Figure 4: Equilibrium Paths with and without Contractionary Monetary Shock

Note: Red lines correspond to reform paths that occur at different points in time.

ical new-Keynesian model.

3.3 Debt stabilization

As our third policy experiment, we consider the case of a monetary authority who attempts to stabilize the government debt. This requires the real rate to be given by $r_t = \rho - \psi$, so $b_t = b_0$. We assume that the output gap is zero at a given date T_0, so the output gap is given by $x_t = \psi(T_0 - t)$. Inflation is given by

$$\pi_t = \frac{\kappa}{\rho + \theta}\left(x_t - \frac{\psi}{\rho + \theta}\right) + \frac{\theta^* \kappa \Phi}{\rho + \theta}(b_0 - b^n). \tag{23}$$

To stabilize debt, the output gap and inflation must have a downward trend. The monetary authority must initially overheat the economy to slow down the accumulation of debt.

3.4 Determinacy and implementability

In the three policy experiments considered so far, we assumed that the monetary authority can implement an equilibrium where the output gap is stabilized, or real rates are exponentially decaying, or government debt is stabilized. We consider next the conditions under which the equilibrium is uniquely determined and whether the monetary authority can implement any given equilibrium.

The equilibrium conditions consist of the Euler equation (10), the NKPC (12), the Taylor rule (3), and the debt dynamics (8). We can write the dynamic system as follows:

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \\ \dot{b}_t \end{bmatrix} = \begin{bmatrix} 0 & \phi - 1 & 0 \\ -\kappa & \rho + \theta_f & -\theta_f^* \kappa \Phi \\ 0 & \phi - 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ b_t \end{bmatrix} + \begin{bmatrix} u_t \\ \theta_f^* \kappa \Phi b^n \\ u_t + \psi_t \end{bmatrix}, \tag{24}$$

given b_0. The next proposition characterizes the conditions for determinacy and how to implement a given equilibrium.

Proposition 4 (Determinacy and implementability)**.** *Consider a given path of monetary disturbances u_t and fiscal shock ψ_t. Then,*

I. *There exists a unique bounded equilibrium if and only if the Taylor principle is satisfied, that is, $\phi > 1$.*

II. *Let $\hat{i}_t$ denote a given path of nominal interest rates and $(\hat{x}_t, \hat{\pi}_t, \hat{b}_t)$ that satisfies the Euler equation (10), the NKPC (12), and the government's flow budget constraint (8). Suppose that $u_t = \hat{i}_t - \rho - \phi\hat{\pi}_t$ such that we can write the policy rule as*

$$i_t = \hat{i}_t + \phi(\pi_t - \hat{\pi}_t). \tag{25}$$

Then, the solution to the system (24) satisfies $x_t = \hat{x}_t$, $\pi_t = \hat{\pi}_t$, and $b_t = \hat{b}_t$.

III. *Consider an undisturbed Taylor rule, i.e. $u_t = 0$, and assume $\psi_t = \psi$. Then, there is a constant inflation rate:*

$$\pi_t = -\frac{\theta^* \Phi}{1 + \theta^* \Phi} \frac{\psi}{\phi - 1}. \tag{26}$$

In turn, the output gap and the path of debt follow:

$$x_t = x_0 - \frac{\theta^* \Phi}{1+\theta^*\Phi}\psi t, \qquad b_t = b_0 + \frac{\psi}{1+\theta^*\Phi} t. \tag{27}$$

Proposition 4 shows that the Taylor principle is necessary and sufficient to guarantee that the equilibrium is (locally) uniquely determined. Moreover, it shows how the monetary authority can implement a given equilibrium by effectively adopting a time-varying inflation target, with a rule $i_t = \hat{i}_t + \phi(\pi_t - \hat{\pi}_t)$.[9] As an example, consider the equilibrium under the output gap stabilization policy. In this case, we want to implement an equilibrium with inflation as given in Proposition 1, that is, the target inflation is $\hat{\pi}_t = \hat{\pi}_0 + \frac{\theta^*\kappa\Phi}{\theta+\rho}\psi t$ and $\hat{\pi}_0 = \frac{\theta^*\kappa\Phi}{\theta+\rho}\left[b_0 - b^n + \frac{\psi}{\theta+\rho}\right]$. As the real rate is equal to ρ, our target for the nominal rate is $\hat{i}_t = \rho + \hat{\pi}_t$. Hence, to implement the output-gap stabilization equilibrium the monetary authority needs to set $u_t = \hat{i}_t - \rho - \phi\hat{\pi}_t = -(\phi-1)\hat{\pi}_t$. Therefore, a negative disturbance to the policy rule is required to stabilize the output gap. The equilibrium under the other two policy experiments can be implemented in a similar manner.

Notice that the equilibrium outcome in Phase II can be implemented using a similar approach by assuming that the monetary authority follows the following policy rule in Phase II: $i_t^* = \rho + \phi\pi_t^* + u_t^*$, given the same Taylor coefficient $\phi > 1$. Therefore, we assume that the disturbances to the Taylor rule are regime-dependent, instead of the coefficients of the policy rule, in contrast to the literature on regime-dependent rules (see e.g. Davig and Leeper, 2007 and Farmer, Waggoner and Zha, 2009).

Proposition 4 also shows the equilibrium implemented by a zero disturbance to the policy rule, $u_t = 0$ for all $t \geq 0$, and a constant fiscal shock $\psi_t = \psi$. Interestingly, this policy engineers a decline in real rates, $r_t - \rho = -\frac{\theta^*\Phi}{1+\theta^*\Phi}\psi$, which slows down the accumulation of debt. The reduction in real rates is stronger the more likely it is the economy will switch to Phase II and the larger is the pass-through from debt to inflation in Phase II. A higher Taylor coefficient ϕ brings inflation closer to zero, but it does not affect the real rate or the debt dynamics.

Why does a simple Taylor rule leads to negative inflation? Suppose, for the sake of argument, that the equilibrium inflation was positive. This implies that the real rate would also be positive, so

[9] For models with a time-varying inflation target see e.g. Ireland (2007) and Cogley and Sbordone (2008).

both government debt and output gap increase over time, creating an upward pressure on inflation. However, higher inflation would lead to higher real rates and even faster increase in output gap and government debt. Inflation would spiral out of control, which is inconsistent with the assumption of a bounded solution. In contrast, it is possible to find a stable outcome with a constant deflation. The output gap decreases over time, while debt increases over time, as the low real rate is not enough to offset the effect of the fiscal expansion. The opposite behavior of the output gap and government debt allows for a stable solution.

4. Soft landing

An important question in the recent inflationary episode following the Covid-19 pandemic regarded the possibility of a *soft landing*, that is, a reduction in inflation without a substantial cost in terms of output or employment. In this section, we study the necessary conditions to achieve a soft landing. We find that expectations effects, from both households and firms, play a significant role in determining whether such an outcome is possible.

4.1 The analytics of soft landing

To capture the idea of a soft landing in a simple way, we say that the monetary authority achieves a soft landing if it is able to reduce inflation while it keeps the output gap constant. Though assuming a constant output gap is certainly an extreme assumption, it enable us to isolate the inflation dynamics caused by expectation effects from the standard response to changes in the output gap.

The challenge of a soft landing. Consider first the case of the standard New Keynesian model. In this case, disinflation is tightly connected to a reduction in output gaps. For instance, suppose that the output gap is exponentially decaying, $x_t = e^{-\theta_x t} x_0$, $x_0 > 0$. A constant output gap corresponds to $\theta_x = 0$. In the standard model, the change in inflation is given by

$$\dot{\pi}_t = \rho \pi_t - \kappa x_t = -\theta_x \pi_t, \tag{28}$$

using $\pi_t = \kappa \frac{x_t}{\rho+\theta_x}$. In this case, disinflation, $\dot{\pi}_t < 0$, requires a declining output gap, $\theta_x > 0$. A constant output gap would lead to constant inflation. The model delivers no soft landing.

Consider next the model discussed in Section 3. It turns out that a soft landing is also not possible in that case. From the discussion of the output-gap stabilization experiment in Section 3.1, a constant output gap would cause a trend in inflation, given our assumption that the fiscal expansion is permanent as long as the economy stays in Phase I. We show next that the same is true even in the case the fiscal expansion is temporary.

Suppose that fiscal transfers are exponentially decaying, i.e., $\psi_t = e^{-\theta_\psi t}\psi_0$. The case discussed in Section 3 corresponds to $\theta_\psi = 0$. Given a constant output gap, the real rate is equal to ρ, so the government debt is given by

$$\dot{b}_t = \psi_t \Rightarrow b_t = b_0 + \frac{1-e^{-\theta_\psi t}}{\theta_\psi}\psi_0. \tag{29}$$

Government debt has now a well-defined long-run level in Phase I, $b_{lr} = b_0 + \frac{\psi_0}{\theta_\psi} > b_0$. Moreover, government debt monotonically increases towards this higher long-run level: $b_t = b_{lr} - \frac{1}{\theta_\psi}\psi_t$.

From the NKPC, we obtain the inflation rate:

$$\pi_t = \frac{\kappa}{\rho+\theta}x_0 + \frac{\theta^*\kappa\Phi}{\rho+\theta}(b_{lr}-b^n) - \frac{1}{\theta_\psi}\frac{\theta^*\kappa\Phi}{\rho+\theta+\theta_\psi}\psi_t. \tag{30}$$

As in the output-gap stabilization experiment, inflation is increasing over time: $\dot{\pi}_t = \frac{\theta^*\kappa\Phi}{\rho+\theta+\theta_\psi}\psi_t > 0$. Inflation does not follow a linear trend, but instead converges to a higher long-run level.

This analysis shows that it is not possible to reduce inflation while keeping the output gap constant. Therefore, a soft landing is also not possible in the model with expectation effects on the firm side only.

Introducing households' expectation effects. We consider next the general case where expectation effects are present for households and firms. In this case, the Euler equation is given by

$$\dot{x}_t = r_t - \rho + \theta_h x_t - \theta_h^* x_t^J.$$

This Euler equation deviates from the textbook one in two important ways. First, the term $\theta_h x_t$ captures the discounting effect of future real rates, as discussed in Section 2.2. Second, the last term captures the effect of the jump in output gap in Phase II. From the expression for the output gap (Equation 15) and the real rate (Equation 14) in Phase II, the jump term in the output gap is given by $x_t^J = b_t - b^n$. To obtain a constant output gap, $x_t = x_0$, the following condition must be satisfied:

$$x_0 = \frac{\theta_h^*}{\theta_h}(b_t - b^n) - \frac{1}{\theta_h}(r_t - \rho),$$

assuming $\theta_h > 0$. In contrast to the case of a standard Euler equation, a constant output gap does not require $r_t = \rho$. If government debt is time-varying, then the real rate must also be time-varying to keep the output gap constant. This can be seen by differentiating the expression above with respect to time:

$$0 = \frac{\theta_h^*}{\theta_h}(r_t - \rho + \psi_t) - \frac{1}{\theta_h}\dot{r}_t \Rightarrow \dot{r}_t = \theta_h^*(r_t - \rho) + \theta_h^*\psi_t, \tag{31}$$

where we used the government's budget constraint. For a given initial condition for the real rate, the differential equation above implies that r_t is given by

$$r_t - \rho = \left(r_0 - \rho + \frac{\theta_h^*\psi_0}{\theta_h^* + \theta_\psi}\right)e^{\theta_h^* t} - \frac{\theta_h^*}{\theta_h^* + \theta_\psi}\psi_t.$$

To stabilize the output gap, the monetary authority must engineer a *reduction* in real rates. Government debt is given by

$$\dot{b}_t = \left(r_0 - \rho + \frac{\theta_h^*\psi_0}{\theta_h^* + \theta_\psi}\right)e^{\theta_h^* t} - \frac{\theta_h^*}{\theta_h^* + \theta_\psi}\psi_t + \psi_t \Rightarrow b_t = b_0 + \left(r_0 - \rho + \frac{\theta_h^*\psi_0}{\theta_h^* + \theta_\psi}\right)\frac{e^{\theta_h^* t} - 1}{\theta_h^*} + \frac{1 - e^{-\theta_\psi t}}{\theta_h^* + \theta_\psi}\psi_0.$$

Hence, for r_0 sufficiently low, government debt is decreasing over time.

Finally, inflation is given by

$$\pi_t = \frac{\kappa}{\rho + \theta_f}x_0 + \frac{\theta_f^*\Phi}{\theta_h^*}\left[\theta_h x_0 + \frac{\left(r_0 - \rho + \frac{\theta_h^*\psi_0}{\theta_h^* + \theta_\psi}\right)e^{\theta_h^* t}}{(\theta_h^* + \theta_\psi)(\rho + \theta_f - \theta_h^*)} - \frac{\psi_t}{(\theta_h^* + \theta_\psi)(\rho + \theta_f)}\right],$$

where $x_0 = \frac{\theta_h^*}{\theta_h}(b_0 - b^n) - \frac{1}{\theta_h}(r_0 - \rho)$, and we assumed $\rho + \theta_f > \theta_h^*$.

Notice that as the output gap is constant, inflation dynamics is driven by the jump inflation term. The next proposition provides the necessary conditions for a soft landing.

Proposition 5 (Soft landing)**.** *Suppose $0 < \theta_h^* < \rho + \theta_f$ and the real interest rate satisfies the differential equation (31). If $r_0 - \rho < -\frac{\theta_h^* \psi_0}{\theta_h^* + \theta_\psi}$, then the output gap is constant and inflation is eventually decreasing. If r_0 is sufficiently low, inflation is decreasing for all $t \geq 0$.*

To achieve a soft landing, a monetary authority must be able to reduce inflation while it maintains the output gap constant. In this case, the reduction in inflation comes exclusively from the jump term. Therefore, real rates must be sufficiently low to bring government debt back to its sustainable level. The reduction in government debt implies that the jump on the output gap also declines over time. Hence, to offset the second term in Equation (11), the real rate must be declining over time. As the present discounted value of jump terms decline over time, the present discounted value of real rates (in absolute value) increases over time, keeping the output gap constant. Expectations effects are crucial to obtain this result. If we were to assume $\theta_h = 0$, then $\dot{x}_t = r_t - \rho$ (recall $\theta_h = 0$ implies $\theta_h^* = 0$) and low rates would cause a declining output gap.

Households' expectations of monetary accommodation, θ_h^*, control how fast real rates must decline, with real rates declining at a slower pace for a smaller θ_h^*. In the limit $\theta_h^* \to 0$, so households believe that the fiscal expansion will be fully resolved by higher future taxes, a soft landing is possible with a constant (negative) real rate. In this case, the present discounted value of real rates is obviously constant, and a negative rate brings down the government debt and jump inflation.

4.2 Disinflation and expectation effects

The possibility of a soft landing illustrates that expectations effects may play a significant role in explaining inflation dynamics in contrast to contemporaneous changes in the output gap. But how important are these expectations effects empirically? Recent evidence suggests these effect may be quite important. Hazell, Herreno, Nakamura and Steinsson (2022) provide evidence that the slope of the Phillips curve is relatively small, and it has been since the 1980s. This indicates that even substantial changes in the output gap may have limited impact on inflation. Moreover, they argue that (transitory) deviations of unemployment from its natural level played a minor role during the Volcker disinflation, with expectation effects explaining the bulk of the change in inflation during that episode.[10]

Hazell et al. (2022) consider a formulation of the NKPC where inflation depends on the present discounted value of the temporary component on unemployment gaps and a term capturing long-term inflation expectations. Our formulation differ from theirs in two important ways. First, given that the economy eventually moves to Phase II or to the steady state with full fiscal support with probability one, long-run inflation expectations are anchored in our economy. Our expectation effects capture short- and medium-run expectations. Second, fluctuations in inflation expectations in their setting are tied to permanent changes in the output gap.

In our setting, are not linked to permanent deviations of output gap or interest rates from its natural level, instead expectation effects are a function of the debt gap $b_t - b^n$. We have focused so far in the case where the natural level of debt is constant and expectation effects are entirely driven by movements in government debt. But this does need to be the case. Suppose that $r_t = \rho$ and $\psi_t = 0$, such that government debt is constant. Let's assume that the natural level of debt, the amount that is fully backed by future taxes, is potentially time varying. In particular, suppose that $b_t^n = (1 - \omega_t)b_0$, so a fraction ω_t of the current level of debt is expected to be wiped out by low rates in Phase II.

The next proposition characterizes the response of inflation to changes in the natural level of debt.

Proposition 6 (Time-varying natural level of debt)**.** *Suppose $r_t = \rho$, $\psi_t = 0$, $b_t^n = (1 - \omega_t)b_0$, where $\omega_t = \omega_0 + \omega_1 e^{-\theta_\omega t}$. Then, the inflation rate is given by*

$$\pi_t = \pi_{lr} + \left[\frac{\theta_h^*}{\theta_h + \theta_\omega} + \theta_f^* \Phi\right] \frac{\kappa \omega_1 e^{-\theta_\omega t} b_0}{\rho + \theta_f + \theta_\omega},$$

where the long-run level of inflation π_{lr} is given by

$$\pi_{lr} \equiv \left[\frac{\theta_h^*}{\theta_h} + \theta_f^* \Phi\right] \frac{\kappa \omega_0 b_0}{\rho + \theta_f}. \tag{32}$$

[10]See also Goodfriend and King (2005) and Bianchi and Ilut (2017) for a related view on the Volcker disinflation.

Proposition 6 shows how fluctuations in the natural level of debt affect inflation. Deviations of debt from its natural level may lead to persistent effects on inflation (conditional on staying in Phase I). The effect on inflation is increasing in the long-run fraction of debt that is unbacked, ω_0, the probability of switching to Phase II perceived by households and firms, θ_h^* and θ_f^*, and it decreasing in the probability of fiscal adjustment perceived by households and firms, $\overline{\theta}_h$ and $\overline{\theta}_f$. Similarly, the temporary component in the fraction of debt that is unbacked affects inflation. A decline in ω_t reduces inflation even if b_t is constant.

4.3 The perils of Taylor rules redux

As discussed above, households' expectation effects create the possibility of achieving a soft landing. However, they also open the door for adverse outcomes when the monetary authority follows an undisturbed Taylor rule, especially when households regard to be unlikely that a fiscal expansion will be followed by higher future taxes.

We focus on the case $\overline{\theta}_h = 0$, so households regard the case of a fiscal adjustment without monetary support unlikely. We interpret this case as capturing the dynamics in developing economies with a history of inflation problems and weak fiscal institutions. One may also assume that $\overline{\theta}_f = 0$, but this is not essential for the results that follow.

Proposition 4 showed that there is a unique bounded equilibrium when the Taylor principle is satisfied for the case without households' expectation effects. The next proposition shows that such effects have profound implications for the equilibrium dynamics.

Proposition 7 (Instability under the Taylor principle)**.** *Suppose $\overline{\theta}_h = 0$ and $\theta_h^* > 0$. Then, generically there exists no bounded solution to the equilibrium conditions when $\phi > 1$. There exists a unique bounded equilibrium when $\phi < 1$.*

Proposition 7 shows a new form of instability generated by a Taylor rule. Benhabib, Schmitt-Grohé and Uribe (2001) has famously shown that a Taylor rule can generate instability through self-fulfilling fluctuations. Here we highlight a different form of instability, where a policy rule satisfying the Taylor principle causes all bounded equilibria to cease to exist. Formally, we show in the appendix that a version of the dynamic system (24) with $\theta_h^* > 0$ has three positive eigenvalues. This implies that for an arbitrary initial condition for debt the system is unstable. The instability holds only generically as there is a unique value of b_0 that delivers a bounded solution. This value depends on the entire sequence of shocks ψ_t, while b_0 is predetermined, so typically there is no mechanism to ensure that b_0 jumps to the value that leads to the stable path.

The instability emphasized here is also different from the one considered by Leeper (1991), and explored by Loyo (1999) to explain the Brazilian hyper-inflation, who considered a case of a simultaneous active monetary and active fiscal regimes. Leeper (1991) shows that there is no bounded solution when both the monetary and fiscal authorities are active and there is no discounting in the Euler equation. In contrast, we focus on the case where the fiscal authority is passive and there is discounting in the Euler equation.[11] The instability in our case comes from the interaction of the monetary rule with expectations effects.

Interestingly, there is no instability when $\phi < 1$. In particular, there is a unique bounded equilibrium even under an interest rate peg, $\phi = 0$. It has been already shown in the literature that discounting in the Euler equation, if strong enough, can cause even an interest rate peg to be determined (see e.g. Acharya and Dogra 2020). In that setting, a policy rule satisfying the Taylor principle is not necessary for determinacy, but it does not create instability if adopted. In our setting, we have a combination of a discounted Euler equation with a jump term on the output gap, which leads to determinacy under an interest rate peg for $\theta_h^* > 0$, even if arbitrarily small. Moreover, the fact that government debt is a relevant state variable implies that the system can be "over-determined," with more positive eigenvalues in the dynamic system than jump variables. This implies that the system will be unstable for nearly all initial conditions for debt.

Intuitively, the instability can be traced back to the feedback loop inflation, real rates, and government debt. High inflation causes the monetary authority to raise rates more than one-to-one, raising real rates. Higher rates pushes up government debt, raising jump inflation and jump output gap, which puts more pressure on inflation.

[11] To see that the fiscal policy is passive, let $\tau_t = \frac{T_t - \overline{T}}{\overline{B}}$ denote the deviations of transfers from steady state and assume the rule $\tau_t = -\gamma b_t + \psi_t$, so debt dynamics is given by $\dot{b}_t = r_t - \rho + (\rho - \gamma) b_t + \psi_t$. The case $\rho - \gamma > 0$ corresponds to active fiscal policy, while we assume that $\gamma = \rho$.

5. Optimal Policy During Phase I

We now study optimal monetary policy during Phase I. We focus on the case $\overline{\theta} = 0$. We consider the following welfare-loss objective function for the government:

$$\mathcal{P} = -\frac{1}{2}\mathbb{E}\left[\int_0^\tau \exp\left(-\rho t\right)\left(\pi_t^2 + \alpha x_t^2\right) dt + \int_\tau^{\tau+T} \exp\left(-\rho t\right)\left(\pi_t^{*2} + \alpha x_t^{*2}\right) dt\right]$$

where τ is the random time of arrival of Phase II. Clearly, since after T the economy is at its steady-state value, the loss only considers losses up to T periods since the start of the reform.

Notation. We define the following auxiliary functions:

$$\mathcal{L}_0^\xi\left(t_1, t_2\right) \equiv \int_{t_1}^{t_2} \exp\left(-\xi t\right) dt, \quad \mathcal{L}_1^\xi\left(t_1, t_2\right) \equiv \int_{t_1}^{t_2} \exp\left(-\xi t\right) t dt, \quad \mathcal{L}_2^\xi\left(t_1, t_2\right) \equiv \int_{t_1}^{t_2} \exp\left(-\xi t\right) t^2 dt.$$

We suppress the arguments when referring to the infinite integration:

$$\mathcal{L}_0^\xi \equiv \int_0^\infty \exp\left(-\xi t\right) dt, \quad \mathcal{L}_1^\xi \equiv \int_0^\infty \exp\left(-\xi t\right) t dt, \quad \mathcal{L}_2^\xi \equiv \int_0^\infty \exp\left(-\xi t\right) t^2 dt.$$

These functions and constants appear multiple times in this section. Their properties are characterized in Appendix B.

Value of Phase II. Consider the value of Phase II, once we have arrived at that phase:

$$\mathcal{P}^{II}\left(b_0^*\right) = \int_0^T \exp\left(-\rho t\right)\left(\pi_t^{*2} + \alpha x_t^{*2}\right) dt.$$

The following Lemma shows that this value is quadratic in the value of debt in that state:

Lemma 2. *The value of Stage II given v_t is:*

$$\mathcal{P}^{II}\left(B_t\right) = \Upsilon^T \left(b_t - b^n\right)^2,$$

where $\Upsilon^T \equiv \left(\alpha\Gamma^T(0) + \kappa^2\Phi^T\right)$ *and* Υ *and* $\Gamma^T(t)$ *are given by:*

$$\Phi^T \equiv \int_0^T \exp(-\rho s)\,\Gamma(s,T)^2\,ds \quad \textit{and} \quad \Gamma^T(t) \equiv \int_t^T \exp(-\rho s)\left(1-\frac{s}{T}\right)^2 ds.$$

While the objective function only encompasses inflation and the output gap, notice how deviation of the debt level relative to its neutral value, ultimately summarize square losses. The value enters into the objective of Phase I. The term $\kappa^2\Phi^T$ captures the cost of inflation in Phase II whereas $\Gamma^T(0)$ captures the cost in terms of the output gap. All in all, Υ^T captures the overall cost of arriving at Phase II with debt different than the neutral debt level. It is a term related to Ψ^T which captures the effect on expected inflation prior to the reform.

Optimal Policy Absent Shocks. Thus, the Phase I value function solves:

$$\mathcal{P} = -\frac{1}{2}\mathbb{E}\left[\int_0^\tau \exp(-\rho t)\left(\pi_t^2 + \alpha x_t^2\right)d\tau + \exp(-\rho\tau)\,\mathcal{P}^{II}(b_\tau)\right],$$

Using the fact that Phase II arrives as a Poisson event, and replacing the value of Phase II, we obtain that the value in phase I can be written as:

Lemma 3. *The value of Stage I is:*

$$\mathcal{P} = -\frac{1}{2}\max_{\{\pi_0,r_t\}}\int_0^\infty \exp\left(-(\theta+\rho)\,t\right)\left(\pi_t^2 + \alpha x_t^2 + \theta\Upsilon^T\left(b_t - b^n\right)^2\right)dt$$

subject to the equilibrium system:

$$\dot{\pi} = (\theta+\rho)\,\pi_t - \kappa x_t - \theta\pi_t^J; \quad \dot{x}_t = r_t - \rho; \quad \dot{b}_t = r_t.$$

The overall objective is akin to the standard losses that emerge in a new-Keynesian model, except that it also quadratic in the level of debt. The quadratic structure is convenient to exploit the linear-quadratic structure of the problem. To simplify the constraint set, we directly work with the output gap as a control and relate the path of log debt (b_t) to the path of the output gap noticing that:

$$b_t = b_0 + x_t - x_0 + \rho t \Longleftrightarrow x_t = x_0 + b_t - b_0 - \rho t.$$

Recall also that jump inflation is given by (16), and hence:

$$\pi_t^J = \kappa \Psi^T (b_t - b_n).$$

Therefore, taking these observations into consideration, we arrive at a simplified problem where the planner chooses a path for the real interest rate.

Lemma 4. *The planner problem in Stage I is given by the following program:*

$$\mathcal{P} = -\frac{1}{2} \max_{\{\pi_0, r_t\}} \int_0^\infty \exp(-(\theta+\rho)t) \left(\pi_t^2 + \alpha (x_0 + b_t - b_0 - \rho t)^2 + \theta \kappa \Upsilon(T) (b_t - b^n)^2 \right) dt$$

subject to:

$$\dot{\pi} = (\theta+\rho)\pi_t - \kappa (x_0 + b_t - b_0 - \rho t) - \theta \kappa \Psi^T (b_t - b^n).$$

$$b_0 \quad given.$$

This simplified problem now casts the entire objective into a maximization problem where the control variable is the real interest rate and an initial condition for inflation. The control problem is subject to a single differential constraint: the Phillips curve.

In turn, the following Lemma shows that the problem can be summarized as choosing a set of linear trends for inflation, the output gap, and debt together with their initial conditions:

Lemma 5. *The optimal inflation, output, and debt are affine functions of time:*

$$\pi_t = c_0^\pi + c_1^\pi t, \quad x_t = c_0^x + c_1^x t, \quad b_t = b_0 + c_1^b t.$$

where the coefficients satisfy the following relationship:

$$\text{(Euler)} \quad c_1^x = c_1^b - \rho$$

$$\text{(Phillips Curve I)} \quad c_1^x = \frac{\theta + \rho}{\kappa + \theta\kappa\Psi^T} c_1^\pi - \rho \frac{\theta\kappa\Psi^T}{\kappa + \theta\kappa\Psi^T}$$

$$\text{(Phillips Curve II)} \quad \rho c_0^\pi = \kappa c_0^x + c_1^\pi + \rho\kappa\Psi^T \left(b_0 - b^n\right)$$

Moreover, $c_1^x = r^*$ has the interpretation of a constant real interest rate and $c_0^\pi = \pi_0$ the initial level of inflation.

The problem boils down to the choice of a number of scalars, which in turn are linked through the dynamic equations of the system. Because the system is affine, we can express the objective function in terms of a solution to a static quadratic problem in $\{\pi_0, r^*\}$. As a result the objective solves a quadratic equation.

Lemma 6. *The value of Stage I is:*

$$\mathcal{P} = \min_{\{\pi_0, r^*\}} \int_0^\infty \exp\left(-\left(\theta + \rho\right) t\right) \left(\left(c_0^\pi + c_1^\pi t\right)^2 + \alpha \left(c_0^x + c_1^x t\right)^2 + \theta\kappa\Upsilon^T \left(b_0 - b^b + c_1^b t\right)^2 \right) dt$$

where:

$$\text{(Euler - Trend)} \quad c_1^x = c_1^b - \rho$$

$$\text{(Phillips Curve - Trend)} \quad c_1^x = \frac{\theta + \rho}{\kappa + \theta\kappa\Psi^T} c_1^\pi - \rho \frac{\theta\kappa\Psi^T}{\kappa + \theta\kappa\Psi^T}$$

$$\text{(Phillips Curve - Constant term)} \quad \rho c_0^\pi = \kappa c_0^x + c_1^\pi + \rho\kappa\Psi^T \left(b_0 - b^n\right)$$

and $c_0^\pi = \pi_0$ and $r^* = c_1^x$.

By substituting the constraints and integrating the terms corresponding to t and t^2 we arrive at static quadratic objective in $\{\pi_0, r^*\}$. This objective yields the optimal solution. Next, we present the key result: the optimal real interest rate target.

Proposition 8 (Optimal Real Rate)**.** *The optimal time zero inflation and real interest rates are:* $\pi_0 = 0$ *and*

$$r^* = \rho - \frac{\left(\kappa^2\left(\theta\Psi^T\left(\frac{\rho}{\theta+\rho}\right)\frac{1+\theta\Psi^T}{\theta+\rho}\right)+\beta\rho\right)\mathcal{L}_2+\beta c_0^b\mathcal{L}_1+\alpha\theta\Psi^T\left(\frac{\rho}{\theta+\rho}-c_0^b\right)\left(\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0+\mathcal{L}_1\right)}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2+\alpha+\beta\right)\mathcal{L}_2-2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1+\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0}.$$

The real interest is implemented through a rule where deviations from the trend in inflation are punished. The required equilibrium rate is:

$$i_t = r^* - c_1^\pi t.$$

Thus, even though real rates are constant, nominal rates also trend. It is useful to verify some benchmarks. For example, if $\theta = 0$, we obtain $r^* = \rho$.

For simplicity, let's assume that the central bank only cares about the cost of inflation, $\alpha = 0$, and $c_0^b = 0$. Then, the optimal rule simplifies to:

$$r^* = \rho\left(1 - \frac{\frac{\theta\Psi^T}{1+\theta\Psi^T}\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2+\beta}{\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2+\beta}\right).$$

In this case, the central bank maintains a real-interest rate below the natural rate of interest, showing that the central bank weights the contemporaneous effect on inflation with the overall effect that debt has on inflation. As θ approaches infinity, the term converges to:

$$r^* = \rho\left(\frac{\left(\Psi^T\right)^2}{\left(\Psi^T\right)^2+\left(\Phi^T\right)^2}\right).$$

This is a weighted average of the ex-ante and ex-post cost.

The example shows that with an expectation of a fiscal reform, attempts to curtail inflation have undesired effects. Any initial increase in rates eventually leads to higher inflation, unlike the canonical new-Keynesian model.

Figure 5 shows the welfare implications of different target real-interest rates, r^*. How far is r^* from ρ depends on the intensity of a reform. For values of 0.04, implying a reform whose time

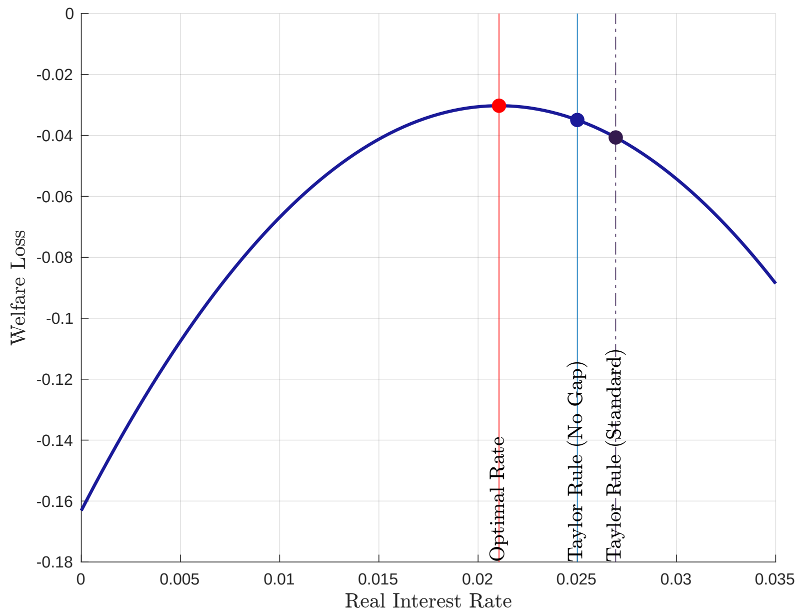


Figure 5: Welfare for Different Policy Rules

Note: Red lines correspond to reform paths that occur at different points in time.

will occur in approximately 25 years, the optimal rate is substantially below the Taylor rules that eliminate the output gap (section 2.1) or the standard rule (section 2.3). The optimal rate allow for a lower rate that eases the growth in the path of debt. The sacrifice comes at the expense of allowing either inflation or the output gap to feature a trend.

6. Conclusion

We have presented a few lessons immediately derived from the model in this paper. First, we demonstrated that in an environment where a fiscal reform is expected to occur, we have shown that attempts to fight inflation backfire through the expectation of greater inflation when the reforms takes place. We called this phenomenon, sticky inflation. Second, we should that because inflation is sticky, optimal policy that cannot guarantee immunity against a forced reform, should balance a trend in inflation with a trend in debt, keeping real interest rates below the natural rate until the reform takes place.

Several policy lessons follow indirectly from the model. First, if a fiscal-monetary reform will happen, it is better to have the reform earlier than later. Second, we have not considered the possibility that earlier attempts to fight inflation are designed to signal that monetary policy will not finance deficits in the future. We have shown that these attempts are futile in bringing inflation back if the signalling effect is not present. Thus, it is important to understand how inflation medium term inflation expectations respond to the signalling effect and to the fiscal effect.

Finally, because the model features trends in inflation or the output gap as outcomes, we consider that an analysis of the non-linear dynamics would be desirable.

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A. Derivation for Section 2

Households. The household problem is given by

$$V_t(B_t) = \max_{[C_s, N_s]_{s \geq t}} \mathbb{E}_t \left[\int_t^{t^*} e^{-\rho(s-t)} \left[u(C_s) - h(N_s) \right] dt + e^{-\rho t} V^*_{t^*}(B^*_t) \right], \tag{33}$$

subject to

$$\dot{B}_t = (i_t - \pi_t) B_t + \frac{W_t}{P_t} N_t + D_t + T_t - C_t, \tag{34}$$

and a No-Ponzi condition, where t^* denotes the arrival time for a Poisson process with intensity $\theta \geq 0$, B_t denotes the real valued of bonds held by households, W_t is the nominal wage, P_t is the price level, D_t are dividends payed by firms, T_t denotes fiscal transfers.

The HJB equation for this problem is given by

$$\rho V = u(C) - h(N) + \dot{V} + V_B \left[(i - \pi) B + \frac{W}{P} N + T - C \right] + \theta [V^* - V]. \tag{35}$$

The first-order conditions are given by

$$u'(C) = V_B, \qquad h'(N) = V_B \frac{W}{P}. \tag{36}$$

The envelope condition is given by

$$\rho V_B = V_B (i - \pi) \dot{V}_B + V_{BB} \left[(i - \pi) B + \frac{W}{P} N + T - C \right] + \theta [V^*_B - V_B]. \tag{37}$$

Combining the envelope condition with the optimality condition for consumption, we obtain

$$0 = (i - \pi - \rho) + \frac{u''(C) C}{C} \frac{\dot{C}_t}{C_t} + \theta \left[\frac{u'(C^*)}{u'(C)} - 1 \right] \Rightarrow \frac{\dot{C}}{C} = \sigma^{-1} (i - \pi - \rho) + \frac{\theta}{\sigma} \left[\frac{u'(C^*)}{u'(C)} - 1 \right], \tag{38}$$

where $\sigma = -\frac{u''(C)C}{u'(C)}$.

The optimality condition for labor can be written as

$$\frac{h'(N)}{u'(C)} = \frac{W}{P}. \tag{39}$$

Firms. There are two types of firms in the economy: final-goods producers and intermediate-goods producers. Final goods are produced by competitive firms according to the production function $Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\epsilon}{\epsilon - 1}} di \right)^{\frac{\epsilon - 1}{\epsilon}}$, where $Y_{i,t}$ denotes the output of intermediate $i \in [0, 1]$. The demand for intermediate i is given by $Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} Y_t$, where $P_{i,t}$ is the price of intermediate i, $P_t = \left(\int_0^1 P_{i,t}^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$ is the price level, and Y_t is the aggregate output.

Intermediate-goods producers have monopoly over their variety and operate the technology $Y_{i,t} = A_t N_{i,t}$, where $N_{i,t}$ denotes labor input. Firms are subject to quadratic adjustment costs on price changes, so the problem of intermediate i is given by

$$Q_{i,t}(P_i) = \max_{[\pi_{i,s}]_{s\geq t}} \mathbb{E}_t \left[\int_t^{t^*} \frac{\eta_s}{\eta_t} \left(\frac{P_{i,s}}{P_{i,t}} Y_{i,s} - \frac{W_s}{P_s} \frac{Y_{i,s}}{A_s} - \frac{\varphi}{2}\pi_{i,s}^2 \right) ds + \frac{\eta_{t^*}}{\eta_t} Q^*_{i,t}(P^*_{i,t}) \right], \tag{40}$$

subject to $Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon} Y_t$ and $\dot{P}_{i,t} = \pi_{i,t} P_{i,t}$, given $P_{i,t} = P_i$ and $\eta_t = e^{-\rho t} u'(C_t)$, where φ is the adjustment cost parameter.

The HJB equation for this problem is

$$0 = \max_{\pi_{i,t}} \eta_t \left(\frac{P_{i,t}}{P_t} Y_{i,t} - \frac{W_t}{P_t} \frac{Y_{i,t}}{A} - \frac{\varphi}{2} \pi_{i,t}^2 \right) dt + \mathbb{E}_t[d(\eta_t Q_{i,t})], \tag{41}$$

where $\frac{\mathbb{E}_t[d(\eta_t Q_{i,t})]}{\eta_t dt} = -(i_t - \pi_t) Q_{i,t} + \frac{\partial Q_{i,t}}{\partial P_{i,t}} \pi_{i,t} P_{i,t} + \frac{\partial Q_{i,t}}{\partial t} + \theta \frac{\eta_t^*}{\eta_t} \left[Q^*_{i,t} - Q_{i,t} \right]$.

The first-order condition is given by

$$\frac{\partial Q_{i,t}}{\partial P_i} P_{i,t} = \varphi \pi_{i,t}.$$

The change in π_t conditional on no switching in state is then given by

$$\left(\frac{\partial^2 Q_{i,t}}{\partial t \partial P_i} + \frac{\partial^2 Q_{i,t}}{\partial P_i^2} \pi_{i,t} P_{i,t} \right) P_{i,t} + \frac{\partial Q_{i,t}}{\partial P_i} \pi_{i,t} P_{i,t} = \varphi \dot{\pi}_{i,t}. \tag{42}$$

The envelope condition with respect to $P_{i,t}$ is given by

$$0 = \left((1-\epsilon) \frac{P_{i,t}}{P_t} + \epsilon \frac{W_t}{P_t A} \right) \left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} \frac{Y_t}{P_{i,t}} + \frac{\partial^2 Q_{i,t}}{\partial t \partial P_i} + \frac{\partial^2 Q_{i,t}}{\partial P_i^2} \pi_{i,t} P_{i,t} + \frac{\partial Q_{i,t}}{\partial P_i} \pi_{i,t} - (i_t - \pi_t) \frac{\partial Q_{i,t}}{\partial P_i} + \theta \frac{\eta_t^*}{\eta_t} \left(\frac{\partial Q^*_{i,t}}{\partial P_i} - \frac{\partial Q_{i,t}}{\partial P_i} \right). \tag{43}$$

Multiplying the expression above by $P_{i,t}$ and using Equation (42), we obtain

$$0 = \left((1-\epsilon) \frac{P_{i,t}}{P_t} + \epsilon \frac{W_t}{P_t A} \right) \left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} Y_t + \varphi \dot{\pi}_t - (i_t - \pi_t) \varphi \pi_{i,t} + \theta \varphi \frac{\eta_t^*}{\eta_t} \left(\pi^*_{i,t} - \pi_{i,t} \right).$$

Rearranging the expression above, we obtain the non-linear New Keynesian Phillips curve

$$\dot{\pi}_t = (i_t - \pi_t) \pi_t + \theta \frac{\eta^*}{\eta_t} (\pi_t - \pi_t^*) - \frac{\epsilon \varphi^{-1}}{A} \left(\frac{W_t}{P_t} - (1 - \epsilon^{-1}) A \right) Y_t.$$

Government and market clearing. The government flow budget constraint is given by

$$\dot{B}_t^g = (i_t - \pi_t) B_t^g + T_t, \tag{44}$$

where B_t^g denotes the real value of government debt. The government must also satisfy the No-Ponzi condition $\lim_{T\to\infty} \mathbb{E}_t[\eta_T B_T^g] = 0$.

The market clearing condition is given by

$$C_t = Y_t, \qquad N_t = \int_0^1 N_{i,t} di, \qquad B_t = B_t^g. \tag{45}$$

B. Useful Integrals

Time Discounting. We define some useful formulas related to some integral terms the re-appear in the body of the paper. The level-zero integral of time discounting is:

$$\int \exp\left(-\rho s\right) ds = -\frac{\exp\left(-\rho s\right)}{\rho}.$$

Thus, define it's indefinite form by:

$$\mathcal{L}^0\left(\rho, t_1, t_2\right) \equiv \int_{t_1}^{t_2} \exp\left(-\rho s\right) ds = \frac{\exp\left(-\rho t_1\right) - \exp\left(-\rho t_2\right)}{\rho}.$$

For the infinite integral we use:

$$\mathcal{L}^0\left(\rho\right) \equiv \int_0^{\infty} \exp\left(-\rho s\right) ds = \frac{1}{\rho}.$$

In addition:

$$\mathcal{L}^0\left(\rho, t_1, \infty\right) = \exp\left(-\rho t_1\right) \mathcal{L}^0\left(\rho\right).$$

Linear-Time Discounting. The time discounting of linear time is:

$$\int \exp\left(-\rho s\right) s ds = -\frac{\exp\left(-\rho s\right)\left(\rho s + 1\right)}{\rho^2}.$$

Thus, it definite form is:

$$\mathcal{L}^1\left(\rho, t_1, t_2\right) \equiv \int_{t_1}^{t_2} \exp\left(-\rho s\right) s ds = \frac{\exp\left(-\rho t_1\right)\left(\rho t_1 + 1\right) - \exp\left(-\rho t_2\right)\left(\rho t_2 + 1\right)}{\rho^2}.$$

For the infinite integral we use:

$$\mathcal{L}^1\left(\rho\right) \equiv \int_0^{\infty} \exp\left(-\rho s\right) ds = \frac{1}{\rho^2}.$$

In addition:

$$\begin{aligned}
\mathcal{L}^1\left(\rho, t_1, \infty\right) &= \exp\left(-\rho t_1\right)\left(\mathcal{L}^0\left(\rho, 0, \infty\right) t_1 + \mathcal{L}^1\left(\rho, 0, \infty\right)\right) \\
&= \mathcal{L}^0\left(\rho, t_1, \infty\right) t_1 + \mathcal{L}^1\left(\rho, 0, \infty\right)
\end{aligned}$$

Quadratic Time. The time discounting of quadratic time is:

$$\int \exp\left(-\rho s\right) s^2 ds = -\frac{\exp\left(-\rho s\right)\left(\rho^2 s^2 + 2\rho s + 2\right)}{\rho^3}.$$

Thus, the definite form is:

$$\begin{aligned}
\mathcal{L}^2\left(\rho, t_1, t_2\right) &\equiv \int_0^{T} \exp\left(-\rho s\right) s^2 ds \\
&= \frac{\exp\left(-\rho t_1\right)\left(\rho^2\left(t_1\right)^2 + 2\rho t_1 + 2\right) - \exp\left(-\rho t_2\right)\left(\rho^2\left(t_1\right)^2 + 2\rho t_2 + 2\right)}{\rho^3}.
\end{aligned}$$

Thus, the integrals of the form:

$$= \int_0^\infty \exp(-\rho s)\, s^2 ds = \frac{2}{\rho^3}.$$

The Integral Ψ^T. This is useful to compute Ψ^T and bound this value. This term is related to:

$$\begin{aligned}
\Psi^T &= \frac{1}{T}\int_0^T \exp(-\rho s)(T-s)\, ds \\
&= \mathcal{L}^0(\rho, 0, T) - \frac{1}{T}\mathcal{L}^1(\rho, 0, T) \\
&= \frac{1-\exp(-\rho T)}{\rho} + \frac{1}{T}\frac{\exp(-\rho T)(\rho T+1)-1}{\rho^2} \\
&= \frac{1}{\rho}\left(1 - \frac{1-\exp(-\rho T)}{T\rho}\right).
\end{aligned}$$

The limits require use to use L'Hospital's rule:

$$\lim_{T\to 0}\Psi^T = \frac{1}{\rho}\left(1 - \lim_{T\to 0}\frac{\rho\exp(-\rho T)}{\rho}\right) = 0.$$

and

$$\lim_{T\to \infty}\Psi^T = \frac{1}{\rho}\left(1 - \lim_{T\to \infty}\frac{\rho\exp(-\rho T)}{\rho}\right) = \frac{1}{\rho}.$$

Furthermore, the term:

$$1 - \frac{1-\exp(-\rho T)}{\rho T} < 1$$

since:

$$1 - \rho T < \exp(-\rho T),$$

which follows from the convexity of the exponential. Thus, $\Psi^T \in [0, \rho]$.

C. Proofs of Section ?? Results

C.1 Proofs in Section ??

Proof of Proposition 1.

Proof.

$$\begin{aligned}
\pi_t &= \theta\int_0^\infty \exp(-(\rho+\theta)z)\left(\kappa\Psi^T(T\rho - \ln(\Delta/\rho) + \ln(B_0\exp(\rho t)) + z)\right) dz \\
&= \theta\int_0^\infty \exp(-(\rho+\theta)z)\left(\kappa\Psi^T(T\rho - \ln(\Delta/\rho) + \ln(B_0) + z)\right) dz + \kappa\Psi^T\theta\rho t\int_0^\infty \exp(-(\rho+\theta)z)\, dz \\
&= \pi_0 + \frac{\rho\kappa\Psi^T}{T}\frac{\theta}{\theta+\rho}t.
\end{aligned}$$

where the first line uses the transformation, $z \equiv s - t$.

$\square$

C.2 Proofs in Section 3.2

Proof of Proposition 2.

Proof. The fight-inflation strategy combats is successful at bringing inflation down if:

$$-F_0^{\pi} > J_0^{\pi}.$$

We can ask how the two effects scale with r^f. In particular, we have that:

$$\frac{\partial J_0^{\pi}}{\partial r^f} = \theta\kappa\Psi^T \left[\int_0^{\infty} \exp\left(-\left(\rho+\theta\right)s\right)\min\left\{s, T_0\right\} ds\right],$$

and for the fight term:

$$\frac{\partial F_0^{\pi}}{\partial r^f} = \kappa \int_0^{T_0} \exp\left(-\left(\rho+\theta\right)s\right)\left(s - T_0\right) ds.$$

Thus, as long as:

$$\frac{\partial J_0^{\pi}}{\partial r^f} + \frac{\partial F_0^{\pi}}{\partial r^f} < 0,$$

there exits a sufficiently high level of r^f such that the policy is deflationary. Breaking the integrals we need:

$$\int_0^{T_0} \exp\left(-\left(\rho+\theta\right)s\right)\left(T_0 - s\right) ds > \theta\Psi^T \left[\int_0^{\infty} \exp\left(-\left(\rho+\theta\right)s\right)\min\left\{s, T_0\right\} ds\right]$$

□

Proof of Proposition ??.

Proof. The left hand side of the condition starts at zero. It's derivative is:

$$\exp\left(-\left(\rho+\theta\right)s\right)\left(T_0 - s\right)|_{s=T_0} + \int_0^{T_0} \exp\left(-\left(\rho+\theta\right)s\right) ds > 0,$$

The term on the right hand side of the condition is positive at $T_0 = 0$. It can be written as:

$$\theta\Psi^T \left[\int_0^{T_0} \exp\left(-\left(\rho+\theta\right)s\right) s ds + T_0 \int_{T_0}^{\infty} \exp\left(-\left(\rho+\theta\right)s\right) ds\right].$$

By Leibniz's rule, the derivative is:

$$\int_{T_0}^{\infty} \exp\left(-\left(\rho+\theta\right)s\right) ds,$$

so it's derivative shrinks. Thus, the term in the left-hand side must be larger for T_0 sufficiently large.

□

Proof of Proposition 3.

Sticky Inflation. The change in the jump term is

$$\frac{\partial J_t^{\pi}}{\partial r^f} = \frac{\kappa\theta\Psi^T}{T} \left[\int_{min\{t,T_0\}}^{T_0} \exp\left(-\left(\rho+\theta\right)s\right) s ds + T_0 \int_{max\{t,T_0\}}^{\infty} \exp\left(-\left(\rho+\theta\right)s\right) ds\right],$$

and for the fight term:

$$\frac{\partial F_0^\pi}{\partial r^f} = \kappa \int_{\min\{t,T_0\}}^{T_0} \exp\left(-\left(\rho+\theta\right)s\right) sds.$$

and

Clearly, there always exists a $t < T_0$ sufficiently large such that inflation comes back because there's always a t such that:

$$\frac{\partial J_t^\pi}{\partial r^f} + \frac{\partial F_0^\pi}{\partial r^f}.$$

This follows directly from continuity.

□

Proof of Proposition 4

Proof. For the general case, the dynamic system is given by

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \\ \dot{b}_t \end{bmatrix} = \begin{bmatrix} \theta_h & \phi-1 & -\theta_h^* \\ -\kappa & \rho+\theta_f & -\theta_f^*\kappa\Phi \\ 0 & \phi-1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ b_t \end{bmatrix} + \begin{bmatrix} u_t + \theta_h^* b^n \\ \theta_f^*\kappa\Phi b^n \\ u_t + \psi_t \end{bmatrix} \tag{46}$$

The equilibrium is uniquely determined if the matrix above has two eigenvalues with positive real components and an eigenvalue with a non-positive real component. The eigenvalues of the system above satisfies the characteristic equation:

$$(\theta_h - \lambda)(\rho+\theta_f-\lambda)(-\lambda) + \theta_h^*\kappa(\phi-1) - \kappa(\phi-1)\lambda + (\theta_h-\lambda)\theta_f^*\Phi\kappa(\phi-1) = 0 \tag{47}$$

The case $\theta_h = \theta_h^*$ has a simple solution. One of the eigenvalues is given by $\lambda_3 = \theta_h$. In this case, the remaining eigenvalues satisfy the condition:

$$\lambda_1 = \frac{\rho+\theta_f + \sqrt{(\rho+\theta_f)^2 - 4\kappa(\phi-1)(1+\theta_f^*\Phi)}}{2}, \qquad \lambda_2 = \frac{\rho+\theta_f - \sqrt{(\rho+\theta_f)^2 - 4\kappa(\phi-1)(1+\theta_f^*\Phi)}}{2}. \tag{48}$$

These two eigenvalues have positive real part if and only if $\phi > 1$. Moreover, the eigenvalues are real-valued if $\phi < 1 + \frac{(\rho+\theta_f)^2}{4\kappa(1+\theta_f^*\Phi)}$. Hence, to obtain only two positive eigenvalues we must impose $\theta_h = 0$. I will then focus on this case. This allow us to reduce the order of the system by writing

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & \phi-1 \\ -\kappa(1+\theta_f^*\Phi) & \rho+\theta_f \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} u_t \\ \theta_f^*\kappa\Phi(b^n + x_0 - b_0 - \hat{\psi}_t) \end{bmatrix}, \tag{49}$$

where $\hat{\psi}_t = \int_0^t \psi_s ds$, using the fact that $b_t = b_0 + x_t - x_0 + \hat{\psi}_t$. The eigenvalues of the system above are λ_1 and λ_2. The matrix of eigenvectors and its inverse are given by

$$V = \begin{bmatrix} 1 & 1 \\ \frac{\lambda_1}{\phi-1} & \frac{\lambda_2}{\phi-1} \end{bmatrix}, \qquad V^{-1} = \frac{\phi-1}{\lambda_2-\lambda_1} \begin{bmatrix} \frac{\lambda_2}{\phi-1} & -1 \\ -\frac{\lambda_1}{\phi-1} & 1 \end{bmatrix}. \tag{50}$$

Let $Z_t = [x_t, \pi_t]'$, $E_t = [u_t, \theta_f^*\kappa\Phi(b^n + x_0 - b_0 - \hat{\psi}_t)]'$, and A denotes the matrix of coefficients, so we can write the dynamic system as $\dot{z}_t = Az_t + B$. The eigendecomposition of $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix with the eigenvalues. Let $z_t = V^{-1}Z_t$ and $e_t = V^{-1}E_t$, then

$$\dot{z}_t = \Lambda z_t + e_t. \tag{51}$$

Solving the two equations forward, we obtain

$$z_{1,t} = -\int_t^\infty e^{-\lambda_1(s-t)} e_{1,s} ds, \qquad z_{2,t} = -\int_t^\infty e^{-\lambda_2(s-t)} e_{2,s} ds. \tag{52}$$

Notice that e_t is given by

$$e_{1,t} = \frac{\lambda_2 u_t - \kappa(\phi-1)\theta_f^*\Phi(b^n + x_0 - b_0 - \hat{\psi}_t)}{\lambda_2 - \lambda_1}, \qquad e_{2,t} = \frac{\kappa(\phi-1)\theta_f^*\Phi(b^n + x_0 - b_0 - \hat{\psi}_t) - \lambda_1 u_t}{\lambda_2 - \lambda_1}. \tag{53}$$

Rotating the system back to the original coordinates, we obtain

$$x_t = z_{1,t} + z_{2,t}, \qquad \pi_t = \frac{\lambda_1 z_{1,t} + \lambda_2 z_{2,t}}{\phi - 1}. \tag{54}$$

The initial value for x_0 satisfies the condition

$$x_0 = \int_0^\infty \left(\frac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) u_t dt + \frac{\kappa(\phi-1)\theta_f^*\Phi}{\lambda_1\lambda_2}(b^n + x_0 - b_0) + \frac{\kappa(\phi-1)\theta_f^*\Phi}{\lambda_1 - \lambda_2} \int_0^\infty \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \hat{\psi}_t dt. \tag{55}$$

Rearranging the expression above, using the fact that $\lambda_1\lambda_2 = \kappa(\phi-1)(1+\theta_f^*\Phi)$, we obtain

$$x_0 = (1+\theta_f^*\Phi) \int_0^\infty \left(\frac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) u_t dt + \theta_f^*\Phi(b^n - b_0) + (1+\theta_f^*\Phi)\frac{\kappa(\phi-1)\theta_f^*\Phi}{\lambda_1 - \lambda_2} \int_0^\infty \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \hat{\psi}_t dt, \tag{56}$$

which is negative if $b^n \leq b_0$ and $\hat{\psi}_t \geq 0$.

The value of x_t is given by

$$x_t = \int_t^\infty \left(\frac{\lambda_2 e^{-\lambda_1(s-t)} - \lambda_1 e^{-\lambda_2(s-t)}}{\lambda_1 - \lambda_2} \right) u_s ds + \frac{\theta_f^*\Phi}{1+\theta_f^*\Phi}(b^n + x_0 - b_0) + \frac{\kappa(\phi-1)\theta_f^*\Phi}{\lambda_1 - \lambda_2} \int_t^\infty \left(e^{-\lambda_1(s-t)} - e^{-\lambda_2(s-t)}\right) \hat{\psi}_s ds. \tag{57}$$

In the special case where $u_t = 0$ and $\psi_t = \psi$, then the output gap is given by

$$x_t = x_0 - \frac{\theta_f^*\Phi}{1+\theta_f^*\Phi}\psi t. \tag{58}$$

Inflation is given by

$$\pi_t = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \int_0^\infty \left(e^{-\lambda_1(s-t)} - e^{-\lambda_2(s-t)}\right) \frac{u_s}{\phi-1} ds + \frac{\kappa(\phi-1)\theta_f^*\Phi}{\lambda_1 - \lambda_2} \int_t^\infty \left(\lambda_1 e^{-\lambda_1(s-t)} - \lambda_2 e^{-\lambda_2(s-t)}\right) \frac{\hat{\psi}_s}{\phi-1} ds. \tag{59}$$

If $u_t = 0$ and $\psi_t = \psi$, then

$$\pi_t = -\frac{\theta_f^*\Phi}{1+\theta_f^*\Phi}\frac{\psi}{\phi-1}. \tag{60}$$

The real rate is given by $i_t - \pi_t = \rho - \frac{\theta_f^* \Phi}{1+\theta_f^* \Phi} \psi$. Government debt evolves according to

$$b_t = b_0 + \frac{\psi}{1 + \theta_f^* \Phi} t. \tag{61}$$

□

C.3 Proofs in Section ??

Proof of Proposition ??.

Proof. Thus, the solution to the path of debt is:

$$B_t = B_0 \exp\left(\rho t + (\phi - 1) \int_0^t \pi_s ds\right).$$

We can also work with

$$b_t = \ln B_t$$

and notice that:

$$\dot{b}_t = r_t$$

We can take time derivatives to the Phillips curve and write it as a second-order differential equation:

$$\ddot{\pi}_t = (\rho + \theta) \dot{\pi}_t - \kappa (\dot{x}_t) + \theta \left(\dot{\pi}_t^J\right).$$

Since we already showed that:

$$\pi_t^J = \pi_0^* (B_t) \equiv \kappa \Psi^T (T\rho - \ln (\Delta/\rho) + \ln B_t),$$

we have that:

$$\dot{\pi}_t^J = \kappa \Psi^T \frac{\dot{B}_t}{B_t} = \kappa \Psi^T (\rho + (\phi - 1) \pi_t).$$

Thus, we obtain that inflation can be written as:

$$\ddot{\pi}_t = (\rho + \theta) \dot{\pi}_t - \kappa \left(\left(1 - \theta \Psi^T\right) (\phi - 1)\right) \pi_t + \theta \rho \kappa \Psi^T.$$

As in the standard new-Keynesian model, inflation can be expressed as a second-order linear differential equation. However, there are some differences. First, the term $\left(1 - \theta\Psi^T\right)(\phi - 1)$ modifies the the usual term of corresponding to the Taylor principle. Even of the condition on the coefficient is the same, and $(\phi - 1)$ is required for determinacy, this version, however, features inhomogeneous part given by the constant term. Thus, even when the Taylor principle is satisfied, which is needed to avoid indeterminacy around the initial conditions, inflation will trend.

Next, we solve for the long-term trend, using the solution trend inflation, using the method of undetermined coefficients. Guess:

$$\pi_t = k_1 \exp(\lambda_1 t) + k_2 \exp(\lambda_2 t) + c_0$$

so that:

$$\dot{\pi}_t = \lambda_1 k_1 \exp(\lambda_1 t) + \lambda_2 k_2 \exp(\lambda_2 t), \quad \ddot{\pi}_t = \lambda_1^2 k_1 \exp(\lambda_1 t) + \lambda_2^2 k_2 \exp(\lambda_2 t).$$

Substituting the solution:

$$\lambda_1^2 k_1 \exp(\lambda_1 t)+\lambda_2^2 k_2 \exp(\lambda_2 t) = (\rho+\theta)(\lambda_1 k_1 \exp(\lambda_1 t) + \lambda_2 k_2 \exp(\lambda_2 t)) - \kappa\left(\left(1-\theta\Psi^T\right)(\phi-1)\right)(k_1 \exp(\lambda_1 t) + k_2 \exp(\lambda_2 t) +$$

Thus, grouping terms we obtain:

$$c_0 = \frac{\theta\kappa\Psi^T}{\kappa\left(\left(1-\theta\Psi^T\right)(\phi-1)\right)}.$$

The term associated with $\exp(\lambda_1 t)$ solves the following:

$$\lambda_1^2 = (\rho+\theta)\lambda_1 - \kappa\left(\left(1-\theta\Psi^T\right)(\phi-1)\right)k_t$$

and finally the terms associated with $k_2 \exp(\lambda_2 t)$ solves the same condition. The corresponding roots are:

$$\{\lambda_1, \lambda_1\} = \frac{1}{2}\left((\rho+\theta) \pm \sqrt{(\rho+\theta)^2 - 4\kappa\left(\left(1-\theta\Psi^T\right)(\phi-1)\right)}\right).$$

Both roots are explosive implies determinacy. In this case, we need that:

$$\left(1-\theta\Psi^T\right)(\phi-1) > 0.$$

They Taylor principle may be reversed provided that

$$1 < \theta\Psi^T.$$

In addition, we obtain that there is trend inflation, even if the conditions for stability are met.

$$\pi_t = \frac{\theta\Psi^T}{\left(\left(1-\theta\Psi^T\right)(\phi-1)\right)}.$$

□

alternative proof. Alternative proof using relation between b_t and x_t. Notice that:

$$\dot{b}_t = r_t$$

Thus:

$$\dot{x}_t = r_t - \rho.$$

Therefore:

$$x_t = b_t + x_0 - b_0 - \rho t.$$

However,

$$\int_0^t r_s ds = (\phi-1)\int_0^t \pi_s ds + \rho t.$$

Thus:

$$b_t = b_0 + (\phi-1)\int_0^t \pi_s ds + \rho t$$

and

$$x_t = (\phi-1)\int_0^t \pi_s ds + x_0.$$

We can take time derivatives to the Phillips curve and write it as a second-order differential equation:

$$\ddot{\pi}_t = (\rho+\theta)\dot{\pi}_t - \kappa(\dot{x}_t) + \theta\left(\dot{\pi}_t^J\right).$$

Since we already showed that:

$$\pi_t^J = \pi_0^* (B_t) \equiv \kappa \Psi^T (T\rho - \ln(\Delta/\rho) + b_t),$$

we have that:

$$\dot{\pi}_t^J = \kappa \Psi^T \frac{\dot{B}_t}{B_t} = \kappa \Psi^T (\rho + (\phi - 1)\pi_t).$$

Thus, we obtain that inflation can be written as:

$$\ddot{\pi}_t = (\rho + \theta)\dot{\pi}_t - \kappa\left(\left(1 - \theta\Psi^T\right)(\phi - 1)\right)\pi_t + \theta\rho\kappa\Psi^T.$$

As in the standard new-Keynesian model, inflation can be expressed as a second-order linear differential equation. However, there are some differences. First, the term $\left(1 - \theta\Psi^T\right)(\phi - 1)$ modifies the the usual term of corresponding to the Taylor principle. Even of the condition on the coefficient is the same, and $(\phi - 1)$ is required for determinacy, this version, however, features inhomogeneous part given by the constant term. Thus, even when the Taylor principle is satisfied, which is needed to avoid indeterminacy around the initial conditions, inflation will trend.

Next, we solve for the long-term trend, using the solution trend inflation, using the method of undetermined coefficients. Guess:

$$\pi_t = k_1 \exp(\lambda_1 t) + k_2 \exp(\lambda_2 t) + c_0$$

so that:

$$\dot{\pi}_t = \lambda_1 k_1 \exp(\lambda_1 t) + \lambda_2 k_2 \exp(\lambda_2 t) + c_1, \quad \ddot{\pi}_t = \lambda_1^2 k_1 \exp(\lambda_1 t) + \lambda_2^2 k_2 \exp(\lambda_2 t).$$

Substituting the solution:

$$\lambda_1^2 k_1 \exp(\lambda_1 t) + \lambda_2^2 k_2 \exp(\lambda_2 t) = (\rho + \theta)(\lambda_1 k_1 \exp(\lambda_1 t) + \lambda_2 k_2 \exp(\lambda_2 t)) - \kappa\left(\left(1 - \theta\Psi^T\right)(\phi - 1)\right)(k_1 \exp(\lambda_1 t) + k_2 \exp(\lambda_2 t) +$$

Thus, grouping terms we obtain:

$$c_0 = \frac{\theta\kappa\Psi^T}{\kappa\left(\left(1 - \theta\Psi^T\right)(\phi - 1)\right)}.$$

The term associated with $\exp(\lambda_1 t)$ solves the following:

$$\lambda_1^2 = (\rho + \theta)\lambda_1 - \kappa\left(\left(1 - \theta\Psi^T\right)(\phi - 1)\right)k_t$$

and finally the terms associated with $k_2 \exp(\lambda_2 t)$ solves the same condition. The corresponding roots are:

$$\{\lambda_1, \lambda_1\} = \frac{1}{2}\left((\rho + \theta) \pm \sqrt{(\rho + \theta)^2 - 4\kappa\left(\left(1 - \theta\Psi^T\right)(\phi - 1)\right)}\right).$$

Both roots are explosive implies determinacy. In this case, we need that:

$$\left(1 - \theta\Psi^T\right)(\phi - 1) > 0.$$

They Taylor principle may be reversed provided that

$$1 < \theta\Psi^T.$$

Thus, in that case, non-explosive roots require $\{k_1, k_2\} = 0$ and thus, inflation is given by the constant:

$$\pi_t = \frac{\theta\Psi^T}{\left(1 - \theta\Psi^T\right)(\phi - 1)}.$$

Next, we verify the path for the output gap and debt. For the output gap:

$$\dot{x}_t = (\phi - 1)\,\pi_t = \frac{\theta\Psi^T}{1-\theta\Psi^T}.$$

Thus, the output gap trends upward:

$$x_t = x_0 + \frac{\theta\Psi^T}{1-\theta\Psi^T}t.$$

In turn, the path for debt follows:

$$b_t = x_t - x_0 + b_0 + \rho t = b_0 + \frac{\theta\Psi^T + \rho\left(1-\theta\Psi^T\right)}{1-\theta\Psi^T}t.$$

□

D. Proofs for Section 5 Results

Proof of Proposition 2.

Stage II Value. In Phase II, we showed that:

$$r^* = \ln\left(B^*_{ss}/B^*_0\right)/T = \ln\left(\Delta/\rho/B^*_0\right)/T$$

$$x^*_t = \left(r^* - \rho\right)(t-T)\,, t \in [0,T]$$

$$\pi^*_t = \left(r^*-\rho\right)\kappa\int_t^T \exp\left(-\rho\,(s-t)\right)(s-T)\,ds.$$

Solving for

$$\int_0^T \exp\left(-\rho t\right)\alpha x^{*2}_t dt = \alpha\left(r^*-\rho\right)^2\Gamma\left(0,T\right)$$

where $\Gamma\left(t,T\right) \equiv \int_t^T \exp\left(-\rho s\right)(T-s)^2\,dt$. In turn:

$$\int_0^T \exp\left(-\rho t\right)\pi^{*2}_t dt = \kappa^2\left(r^*-\rho\right)^2\Upsilon\left(T\right).$$

$$\Upsilon\left(T\right) \equiv \int_0^T \exp\left(-\rho s\right)\Gamma\left(s,T\right)^2 ds.$$

Thus, we can write:

$$\mathcal{P}^{II}\left(B^*_0\right) \quad = \quad \left(r^*-\rho\right)^2\left(\alpha\Gamma\left(0,T\right) + \kappa^2\Upsilon\left(T\right)\right).$$

Hence:

$$\mathcal{P}^{II}\left(B_t\right) = \left(\left(\ln\left(\Delta/\rho\right) - \ln\left(B_t\right)\right)/T - \rho\right)^2\left(\alpha\Gamma\left(0,T\right) + \kappa^2\Upsilon\left(T\right)\right).$$

□

Proof of Proposition 3.

Objective Function. Using the density of a Poisson event $\theta \exp(-\theta\tau)$ we can write the objective as:

$$\mathcal{P} = \int_0^\infty \theta \exp(-\theta\tau) \int_0^\tau \exp(-\rho t)\left(\pi_t^2 + \alpha x_t^2\right) dt d\tau + \theta \exp(-(\rho+\theta)\tau)\,\mathcal{P}^{II}(B_\tau)\, d\tau$$

...

apply Leibnitz's rule to obtain that the first term is:

$$\int_0^\infty \theta \exp(-\theta\tau) \int_0^\tau \exp(-\rho t)\left(\pi_t^2 + \alpha x_t^2\right) dt d\tau = -\exp(-\theta\tau) \int_0^\tau \exp(-\rho t)\left(\pi_t^2 + \alpha x_t^2\right) dt|_{\tau=0}^{\infty} + \exp(-\theta\tau)\exp(-\rho\tau)\left(\pi_\tau^2 + \alpha x\right.$$

Thus, the objective is:

$$\mathcal{P} = \int_0^\infty \exp(-(\theta+\rho)t)\left(\left(\pi_t^2 + \alpha x_t^2\right) + \theta\mathcal{P}^{II}(B_t)\right) dt.$$

We setup a Hamiltonian:

$$\mathcal{P} = \max_{\{r_t\}} \int_0^\infty \exp(-(\theta+\rho)t)\left(\left(\pi_t^2 + \alpha x_t^2\right) + \theta\mathcal{P}^{II}(B_t)\right) dt$$

subject to:

$$\dot{\pi} = (\theta+\rho)\pi_t - \kappa x_t - \theta\pi_t^J$$

$$\dot{x}_t = r_t - \rho$$

$$\dot{B}_t = r_t B_t.$$

There are two differences with a standard new-Keynesian optimization. First, jump inflation depends on debt. Second, the value of debt is there. Three states and one control variable. It is easier to work with the logarithm of B_t since in that case, the objective is entirely linear quadratic. In that case...

□

Proof of Proposition 7.

Hamiltonian System. The optimization conditions for the system are:

$$\mathcal{H}_b = 0, \quad \mathcal{H}_\pi = -\dot{\lambda}_t^\pi.$$

Therefore, we have:

$$(\mathcal{H}_b = 0): \quad \alpha(x_0 + b_t - b_0 - \rho t) - \theta\left(\left(\ln(\Delta/\rho) - b_t\right)/T - \rho\right)\Upsilon(T)/T = \lambda_t^\pi \kappa\left(1 + \theta\Psi^T\right).$$

Thus, the system is augment by the effect on debt. The system provides a linear relation between the co-state and control, debt.

Next have that:

$$\left(\mathcal{H}_\pi = -\dot{\lambda}_t^\pi\right): \quad \pi_t(\theta+\rho) + \lambda_t^\pi = -\dot{\lambda}_t^\pi.$$

Finally, the system is closed with the Phillips curve:

$$\dot{\pi}_t = (\theta+\rho)\pi_t - \kappa(x_0 + b_t - b_0 - \rho t) - \theta\kappa\Psi^T\left(T\rho - \ln(\Delta/\rho) + b_t\right).$$

We can clear b_t to obtain a mapping from the co-state to the control:

$$b_t = \frac{\lambda_t^\pi \kappa\left(1 + \theta\Psi^T\right) + \alpha(b_0 + \rho t - x_0) + \theta\left(\left(\ln(\Delta/\rho)\right)/T - \rho\right)\Upsilon(T)/T}{\left(\alpha + \theta\Upsilon(T)/T^2\right)}.$$

Thus, the optimal path of debt satisfies:

$$b_t = \beta_0 + \beta_1 t + \beta_\lambda \lambda_t^\pi .$$

where:

$$\beta_0 = \frac{\alpha\left(b_0 - x_0\right) + \theta\left(\left(\ln\left(\Delta/\rho\right)\right)/T - \rho\right)\Upsilon\left(T\right)/T}{\left(\alpha + \theta\Upsilon\left(T\right)/T^2\right)}$$

$$\beta_1 = \frac{\alpha\rho}{\alpha + \theta\Upsilon\left(T\right)/T^2}$$

$$\beta_\lambda = \kappa\frac{1 + \theta\Psi^T}{\alpha + \theta\Upsilon\left(T\right)/T^2}.$$

With this representation, we obtain a simple solution to the path of inflation and the co-state as a system:

$$\dot{\lambda}_t^\pi = -\left(\theta + \rho\right)\pi_t - \lambda_t^\pi$$

and

$$\dot{\pi}_t = \left(\theta + \rho\right)\pi_t - \kappa\left(x_0 + \beta_0 + \beta_1 t + \beta_\lambda\lambda_t^\pi - b_0 - \rho t\right) - \theta\kappa\Psi^T\left(T\rho - \ln\left(\Delta/\rho\right) + \beta_0 + \beta_1 t + \beta_\lambda\lambda_t^\pi\right).$$

Grouping terms:

$$\dot{\pi}_t = \left(\theta + \rho\right)\pi_t - \kappa\left(1 + \theta\Psi^T\right)\beta_\lambda\lambda_t^\pi - \kappa\left(x_0 - b_0 + \left(1 + \theta\Psi^T\right)\beta_0 + \theta\Psi^T\left(T\rho - \ln\left(\Delta/\rho\right)\right)\right) - \kappa\left(\left(1 + \theta\Psi^T\right)\beta_1 - \rho\right)t.$$

□

Hamiltonian System. Thus, the planner's problem is equivalent to choosing a path for government debt. This system implies that we can solve the problem as a simple Hamiltonian with one control, one state. The complication is that the Hamiltonian is no longer stationary. Still we can set it up in present value:

$$\begin{aligned}\mathcal{H} &\equiv \frac{1}{2}\left(\pi_t^2 + \alpha\left(x_0 + b_t - b_0 - \rho t\right)^2 + \theta\left(b_t - b^n\right)^2\Upsilon\left(T\right)\right)\\ &+ \lambda_t^\pi\left(\left(\theta + \rho\right)\pi_t - \kappa\left(x_0 + b_t - b_0 - \rho t\right) - \theta\kappa\Psi^T\left(b_t - b^n\right)\right).\end{aligned}$$

The solution to the Hamiltonian system is characterized through the following proposition:

Lemma 7. *The optimal path for inflation satisfies the following optimization system:*

$$\dot{\lambda}_t^\pi = -\left(\theta + \rho\right)\pi_t - \lambda_t^\pi$$

$$\dot{\pi}_t = \left(\theta + \rho\right)\pi_t - \gamma^\pi\lambda_t^\pi - \gamma_0 - \gamma_1 t.$$

Notice that the equation for inflation inherits a linear trend. Thus, inflation will grow at a constant rate. Unlike the Taylor rule, which is successful at taming inflation, at the expense of an exploding debt path, the optimal path of inflation weights the cost of inflation against the cost of exploding debt and an ever growing output gap.

Proof of Lemma 5.

Proof. Part 1.

In the first part of the proof, we show that optimal inflation is affine in time.

Taking time derivatives to the equation for co-state, we obtain:

$$\ddot{\lambda}_t^\pi = -\left(\theta + \rho\right)\dot{\pi}_t - \dot{\lambda}_t^\pi$$

and from there, we can replace the Phillips curve:

$$\ddot{\lambda}_t^\pi = -(\theta+\rho)\left((\theta+\rho)\pi_t - \gamma^\pi \lambda_t^\pi - \gamma_0 - \gamma_1 t\right) - \dot{\lambda}_t^\pi.$$

Inflation appears in levels, but we clear it using the equation for the co-state:

$$\ddot{\lambda}_t^\pi = -(\theta+\rho)\left(\dot{\lambda}_t^\pi - \lambda_t^\pi - \gamma^\pi \lambda_t^\pi - \gamma_0 - \gamma_1 t\right) - \dot{\lambda}_t^\pi.$$

Thus, we obtain:

$$\ddot{\lambda}_t^\pi = -(\theta+\rho+1)\dot{\lambda}_t^\pi + (\theta+\rho)(1+\gamma^\pi)\lambda_t^\pi + (\theta+\rho)(\gamma_0+\gamma_1 t).$$

The roots for the co-state equation are:

$$\beta_{1,2} = \frac{1}{2}\left[(\theta+\rho+1) \pm \sqrt{(\theta+\rho+1)^2 - 4(\theta+\rho)(1+\gamma^\pi)}\right].$$

The particular solution is:

$$\lambda_t^p = C_0 + C_1 t$$

In which case:

$$C_1 = -\frac{\gamma_1}{1+\gamma^\pi}$$

$$-(\theta+\rho+1)C_1 + (1+\gamma^\pi)(\theta+\rho)C_0 = -(\theta+\rho)\gamma_0$$

Thus:

$$C_0 = -\left(\frac{\gamma_0}{1+\gamma^\pi} + \frac{\gamma_1}{1+\gamma^\pi}\frac{(\theta+\rho+1)}{(1+\gamma^\pi)(\theta+\rho)}\right)$$

The particular solution fluctuates around:

$$\lambda_t^p = -\left(\frac{\gamma_0}{(1+\gamma^\pi)} + \frac{\gamma_1}{1+\gamma^\pi}\frac{(\theta+\rho+1)}{(1+\gamma^\pi)(\theta+\rho)} + \frac{\gamma_1}{1+\gamma^\pi}t\right).$$

The general solution satisfies:

$$\lambda_t^c = k_0 \exp(\beta_1 t) + k_1 \exp(\beta_2 t).$$

If we work with non-imaginary roots, both roots are explosive. In that case, we have that: $\{k_0, k_1\} = 0$.

The solution is thus given by:

$$\lambda_t^\pi = \lambda_t^p.$$

Hence:

$$C_1 = -(\theta+\rho)\pi_t - (C_0 + C_1 t)$$

Thus:

$$\pi_t = -(C_1 + C_0) - C_1 t$$

Thus:

$$\pi_t = \frac{1}{1+\gamma^\pi}\left(\gamma_0 + \gamma_1\left(\frac{\theta+\rho+1}{(1+\gamma^\pi)(\theta+\rho)} + 1\right)\right) + \frac{\gamma_1}{1+\gamma^\pi}t.$$

Then, the path of the output gap satisfies:

$$\kappa x_t = (\theta+\rho)\pi_t - \dot{\pi} - \theta\pi_t^J = (\theta+\rho)\pi_t - \dot{\pi} - \theta\pi_t^J$$

From here we derive the real interest rate and the path for debt.

Part 2.

After discarding the explosive roots, we modify use the affine structure to relate to the coefficients.

$\square$

Next,

Solution. We expand the quadratic terms in the objective function after we replace the affine structure of the solution. We obtain the following objective function:

$$\mathcal{P} = \min_{\{\pi_0, r^*\}} \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right)\left(\left(c_0^\pi + c_1^\pi t\right)^2 + \alpha\left(c_0^x + c_1^x t\right)^2 + \beta\left(c_0^b + c_1^b t\right)^2\right)dt$$

where:

$$c_0^b \equiv b_0 - b^n$$

$$\beta \equiv \theta \Upsilon^T$$

We expand the quadratic term:

$$\min_{\{\pi_0, r^*\}} \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right)\left(\left(c_0^\pi\right)^2 + 2c_0^\pi c_1^\pi t + \left(c_1^\pi\right)^2 t^2 + \alpha\left(\left(c_0^x\right)^2 + 2c_0^x c_1^x t + \left(c_1^x\right)^2 t^2\right) + \beta\left(\left(c_0^b\right)^2 + 2c_0^b c_1^b t + \left(c_1^b\right)^2 t^2\right)\right)dt$$

Following the previous Lemma, we have the following set of constraints among the coefficients.

First, we use the following identities:

$$c_1^x = r^*$$

and

$$c_0^\pi = \pi_0.$$

I) From the Euler equation:

$$\dot{x} = r^* - \rho = c_1^b - \rho,$$

and thus:

$$c_1^b = c_1^x - \rho = r^*.$$

II) Using the linear trend components of the Phillips curve:

$$\left(\theta+\rho\right)c_1^\pi = \left(\kappa c_1^x + \theta\kappa\Psi^T c_1^b\right),$$

thus, regrouping and using the Euler equation relationship (xxx):

$$c_1^\pi = \kappa\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \kappa\frac{\rho}{\theta+\rho}.$$

Then, the linear terms are related via:

$$\left(\theta+\rho\right)c_0^\pi = \kappa c_0^x + c_1^\pi + \theta\kappa\Psi^T c_0^b.$$

Clearing c_0^x yields:

$$c_0^x = \frac{\left(\theta+\rho\right)}{\kappa}\pi_0 - \frac{1}{\kappa}c_1^\pi - \theta\Psi^T c_0^b.$$

Substituting the expression for c_1^π we obtain:

$$c_0^x = \frac{\theta+\rho}{\kappa}\pi_0 - \frac{1+\theta\Psi^T}{\theta+\rho}r^* + \frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b$$

Thus, we arrive at:

$$c_0^x = \frac{\theta+\rho}{\kappa}\pi_0 - \frac{1+\theta\Psi^T}{\theta+\rho}r^* + \frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b.$$

$$c_0^b = b_0 - b^b$$

$$c_1^b = r^*$$

$$c_0^\pi = \pi_0$$

$$c_1^\pi = \kappa\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \kappa\frac{\rho}{\theta+\rho}.$$

After the corresponding substitutions, we arrive at a quadratic system.

- Argue that since c_0^x increasing in c_0^π, ideally, c_0^π is minimized provided that $c_0^x > 0$, which must be verified. This assumption reduces the calculation burden.

When $c_0^\pi = 0$, the simplified system is:

$$\min_{\{r^*\}} \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right)\left(\left(c_1^\pi\right)^2 t^2 + \alpha\left(\left(c_0^x\right)^2 + 2c_0^x c_1^x t + \left(c_1^x\right)^2 t^2\right) + \beta\left(\left(c_0^b\right)^2 + 2c_0^b c_1^b t + \left(c_1^b\right)^2 t^2\right)\right) dt$$

subject to:

$$c_0^x = -\frac{1+\theta\Psi^T}{\theta+\rho}r^* + \frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b.$$

$$c_0^b = b_0 - b^b$$

$$c_1^b = r^*$$

$$c_1^\pi = \kappa\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \kappa\frac{\rho}{\theta+\rho}.$$

Thus, after the corresponding substitutions we arrive at:

$$\frac{1}{2}\min_{\{r^*\}} \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right)\left(\left(c_1^\pi\right)^2 t^2 + \alpha\left(\left(c_0^x\right)^2 + 2c_0^x c_1^x t + \left(c_1^x\right)^2 t^2\right) + \beta\left(\left(c_0^b\right)^2 + 2c_0^b c_1^b t + \left(c_1^b\right)^2 t^2\right)\right) dt$$

Let

$$\mathcal{L}_2^{\theta+\rho} \equiv \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right) t^2 dt$$

$$\mathcal{L}_1^{\theta+\rho} \equiv \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right) t dt$$

$$\mathcal{L}_0^{\theta+\rho} \equiv \int_0^\infty \exp\left(-\left(\theta+\rho\right)t\right) dt$$

passing integrals we obtain:

$$\frac{1}{2}\beta\left(c_0^b\right)^2\mathcal{L}_0^{\theta+\rho} + \min_{\{r^*\}}\frac{1}{2}\left(\left(c_1^\pi\right)^2 + \alpha\left(c_1^x\right)^2 + \beta\left(c_1^b\right)^2\right)\mathcal{L}_2 + \left(\alpha c_0^x c_1^x + \beta c_0^b c_1^b\right)\mathcal{L}_1 + \frac{\alpha}{2}\left(c_0^x\right)^2\mathcal{L}_0$$

subject to:

$$c_0^x = -\frac{1+\theta\Psi^T}{\theta+\rho}r^* + \frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b.$$

$$c_0^b = b_0 - b^b$$

$$c_1^b = r^*$$

$$c_1^\pi = \kappa \frac{1+\theta\Psi^T}{\theta+\rho} r^* - \kappa\frac{\rho}{\theta+\rho}.$$

Using our the coefficients:

$$\min_{\{r^*\}} \frac{1}{2}\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \kappa\frac{\rho}{\theta+\rho}\right)^2 + \alpha\left(r^*-\rho\right)^2 + \beta r^{*2}\right)\mathcal{L}_2 + \left(\alpha\left(-\frac{1+\theta\Psi^T}{\theta+\rho}r^* + \frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b\right)\left(r^*-\rho\right) + \beta c_0^b r^*\right)\mathcal{L}_1 + \frac{\alpha}{2}$$

Next, we take first-order conditions:

$$\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \kappa\frac{\rho}{\theta+\rho}\right)\frac{1+\theta\Psi^T}{\theta+\rho}\kappa + \alpha\left(r^*-\rho\right) + \beta r^*\right)\mathcal{L}_2 + \left(\alpha\left(-2\frac{1+\theta\Psi^T}{\theta+\rho}r^* - \theta\Psi^T c_0^b + \frac{2+\theta\Psi^T}{\theta+\rho}\rho\right) + \beta c_0^b\right)\mathcal{L}_1 + \alpha\left(\frac{1}{}\right.$$

Next, we collect the terms associated with r^*. We obtain:

$$\left(\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0\right)r^* = \left(\kappa^2\frac{\rho}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho} + \alpha\rho\right)\mathcal{L}_2 + \left(\alpha\left(\theta\Psi^T c_0^b -\right.\right.$$

Thus, re-arraning terms, we obtain:

$$r^* = \frac{\left(\kappa\frac{\rho}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho}\kappa + \alpha\rho\right)\mathcal{L}_2 + \left(\alpha\left(\theta\Psi^T c_0^b - \frac{2+\theta\Psi^T}{\theta+\rho}\rho\right) - \beta c_0^b\right)\mathcal{L}_1 + \alpha\left(\frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b\right)\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0}.$$

Then,

$$\begin{aligned}
r^* &= \rho + \frac{\left(\kappa\frac{\rho}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho}\kappa + \alpha\rho\right)\mathcal{L}_2 + \left(\alpha\left(\theta\Psi^T c_0^b - \frac{2+\theta\Psi^T}{\theta+\rho}\rho\right) - \beta c_0^b\right)\mathcal{L}_1 + \alpha\left(\frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b\right)\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0} - \rho \\
&= \rho + \frac{-\rho\left(\kappa^2\frac{\theta\Psi^T}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho} + \beta\right)\mathcal{L}_2 + \left(\alpha\left(\theta\Psi^T c_0^b + \frac{\theta\Psi^T}{\theta+\rho}\rho\right) - \beta c_0^b\right)\mathcal{L}_1 + \left(\alpha\frac{-\rho}{\theta+\rho} - \alpha\rho\frac{1+\theta\Psi^T}{\theta+\rho} - \alpha\theta\Psi^T c_0^b\right)\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0} \\
&= \rho + \frac{-\rho\left(\kappa^2\frac{\theta\Psi^T}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho} + \beta\right)\mathcal{L}_2 + \left(\alpha\frac{\theta\Psi^T}{\theta+\rho}\rho + \left(\alpha\theta\Psi^T - \beta\right)c_0^b\right)\mathcal{L}_1 + \alpha\left(\frac{-2\rho-\rho\theta\Psi^T}{\theta+\rho} - \theta\Psi^T c_0^b\right)\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0} \\
&= \rho\left(1 - \frac{\left(\kappa^2\frac{\theta\Psi^T}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho} + \beta\right)\mathcal{L}_2 + \alpha\left(\frac{2+\theta\Psi^T}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_1 - \theta\Psi^T\mathcal{L}_1\right)\frac{1}{\theta+\rho} + \left(\left(\beta - \alpha\theta\Psi^T\right)\mathcal{L}_1 + \alpha\theta\Psi^T\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0\right)\frac{c_0^b}{\rho}}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0}\right) \\
&= \rho\left(1 - \frac{\left(\kappa^2\frac{\theta\Psi^T}{\theta+\rho}\frac{1+\theta\Psi^T}{\theta+\rho} + \beta\right)\mathcal{L}_2 + \alpha\left(\left(2 + \left(\theta\Psi^T\right)^2\right)\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0 - \theta\Psi^T\mathcal{L}_1\right)\frac{1}{\theta+\rho} + \left(\left(\beta - \alpha\theta\Psi^T\right)\mathcal{L}_1 + \alpha\theta\Psi^T\frac{1+\theta\Psi^T}{\theta+\rho}\mathcal{L}_0\right)\frac{c}{\rho}}{\left(\left(\kappa\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \alpha + \beta\right)\mathcal{L}_2 - 2\alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)\mathcal{L}_1 + \alpha\left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2\mathcal{L}_0}\right.
\end{aligned}$$

Let $\theta = 0$, then, using that $\beta = 0$, we obtain:

$$r^* = \rho \left(1 - \frac{\left(\frac{\kappa^2}{\rho^2}\right) \mathcal{L}_2 + \frac{2\alpha}{\rho^2} \mathcal{L}_0}{\left(\left(\frac{\kappa^2}{\rho^2}\right) + \alpha\right) \mathcal{L}_2 - 2\frac{\alpha}{\rho} \mathcal{L}_1 + \alpha \left(\frac{1}{\rho}\right)^2 \mathcal{L}_0} \right) = \rho \left(1 - \frac{1}{1/\rho} \left(\frac{\left(\kappa^2/\rho^2\right) \mathcal{L}_2 + 2\alpha \mathcal{L}_0/\rho}{\left(\left(\frac{\kappa^2}{\rho}\right) + \alpha\rho\right) \mathcal{L}_2 - 2\alpha \mathcal{L}_1 + \alpha \left(\frac{1}{\rho}\right) \mathcal{L}_0} \right) \right) = \rho.$$

This verifies the standard solution that there is no trend in inflation.

Consider no weight on the output gap, $\alpha = 0$. Then:

$$r^* = \frac{\left(\kappa \frac{\rho}{\theta+\rho} \frac{1+\theta\Psi^T}{\theta+\rho} \kappa\right) \mathcal{L}_2 - \beta c_0^b \mathcal{L}_1}{\left(\left(\kappa \frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 + \beta\right) \mathcal{L}_2}.$$

Next, consider $\kappa = 0$ as if the Phillips curve is flat:

$$r^* = \frac{\alpha\rho\mathcal{L}_2 + \left(\alpha \left(\left(\theta\Psi^T c_0^b - \frac{\rho}{\theta+\rho}\right) - \frac{1+\theta\Psi^T}{\theta+\rho}\rho\right) - \beta c_0^b\right) \mathcal{L}_1 + \alpha \left(\frac{\rho}{\theta+\rho} - \theta\Psi^T c_0^b\right) \frac{1+\theta\Psi^T}{\theta+\rho} \mathcal{L}_0}{(\alpha+\beta)\mathcal{L}_2 - 2\alpha \left(\frac{1+\theta\Psi^T}{\theta+\rho}\right) \mathcal{L}_1 + \alpha \left(\frac{1+\theta\Psi^T}{\theta+\rho}\right)^2 \mathcal{L}_0}.$$

□

E. Optimal policy

Value of Phase II. The value of Phase II, conditional on starting with debt level b_0^*, is given by

$$\mathcal{P}^{II}(b_0^*) = \int_0^{T^*} e^{-\rho t}(\alpha x_t^{*2} + \beta \pi_t^{*2}) dt. \tag{62}$$

Using the fact that $x_t^* = (b_0^* - b^n)\left(1 - \frac{t}{T^*}\right)$ and $\pi_t^* = \kappa\Phi(b_0^* - b^n)\left(1 - \frac{t}{T^*}\right)$, we obtain

$$\mathcal{P}^{II}(b_0^*) = \Upsilon(b_0^* - b^n)^2, \tag{63}$$

where $\Upsilon \equiv \left[\alpha + \beta(\kappa\Phi)^2\right] \int_0^{T^*} e^{-\rho t} \left(1 - \frac{t}{T^*}\right)^2 dt$.

The optimal policy problem. The planner's objective can be written as

$$\mathcal{P} = -\frac{1}{2}\mathbb{E}\left[\int_0^{\tau} e^{-\rho t}\left(\alpha x_t^2 + \beta\pi_t^2\right) dt + e^{-\rho\tau}\tilde{\mathcal{P}}_\tau\right],$$

where τ denotes the random time at which the economy switches to either Phase II or the steady state, and $\tilde{\mathcal{P}}_t$ denotes the value after the economy switches to either state. If the economy goes to steady state, then $\tilde{\mathcal{P}}_\tau = 0$, the value in steady state, and if the economy's go to Phase II, then $\tilde{\mathcal{P}}_\tau = \mathcal{P}^{II}(b_\tau)$. The density of τ is $\theta e^{-\theta\tau}$ and, conditional on τ, the economy switches to Phase II with probability $\frac{\theta^*}{\theta}$ and to steady state with the remaining probability (see e.g. Cox and Miller (1977) for a derivation). We can then write the expression above as follows

$$\begin{aligned}
\mathcal{P} &= -\frac{1}{2}\int_0^\infty \theta e^{-\theta\tau}\left[\int_0^\tau e^{-\rho t}\left(\alpha x_t^2 + \beta\pi_t^2\right) dt + e^{-\rho\tau}\frac{\theta^*}{\theta}\mathcal{P}^{II}(b_\tau)\right] d\tau \\
&= -\frac{1}{2}\int_0^\infty e^{-(\rho+\theta)t}\left[\alpha x_t^2 + \beta\pi_t^2 + \theta^*\Upsilon(b_t - b^n)^2\right] dt.
\end{aligned}$$

The planner faces the following implementability constraints:

$$\dot{\pi}_t = (\rho + \theta_f)\pi_t - \kappa x_t - \theta_f^* \kappa \Phi(b_t - b^n), \qquad \dot{x}_t = r_t - \rho + \theta_h x_t - \theta_h^*(b_t - b^n), \qquad \dot{b}_t = r_t - \rho + \psi_t, \tag{64}$$

given the initial condition b_0.

We consider first the case without households' expectation effects: $\theta_h = \theta_h^* = 0$. We further assume that $\theta = \theta_f$ and $\theta^* = \theta_f^*$, so the planner's beliefs coincide with the firm's beliefs. In this case, the output gap is given by $x_t - x_0 = b_t - b_0 - \hat{\psi}_t$, where $\hat{\psi}_t = \int_0^t \psi_s ds$.

We can then write the planner's problem as follows:

$$\max_{\{[\pi_t, b_t, x_t, r_t]_0^\infty\}} -\frac{1}{2} \int_0^\infty e^{-(\rho+\theta)t} \left[\alpha x_t^2 + \beta \pi_t^2 + \theta^* \Upsilon (b_t - b^n)^2\right] dt, \tag{65}$$

subject to

$$\dot{\pi}_t = (\rho + \theta)\pi_t - \kappa x_t - \theta^* \kappa \Phi(b_t - b^n) \tag{66}$$

$$\dot{b}_t = r_t - \rho + \psi_t \tag{67}$$

$$\dot{x}_t = r_t - \rho, \tag{68}$$

given b_0.

Optimality conditions. The Hamiltonian to this problem is given by

$$\mathcal{H} = -\frac{1}{2}\left[\alpha x_t^2 + \beta \pi_t^2 + \theta^* \Upsilon (b_t - b^n)^2\right] + \mu_{\pi,t}\left[(\rho+\theta)\pi_t - \kappa x_t - \theta^* \kappa \Phi(b_t - b^n)\right] + \mu_{b,t}\left[r_t - \rho + \psi_t\right] + \mu_{x,t}[r_t - \rho], \tag{69}$$

The dynamics of the co-state on inflation is given by

$$\dot{\mu}_{\pi,t} - (\rho+\theta)\mu_{\pi,t} = \beta \pi_t - \mu_{\pi,t}(\rho + \theta), \tag{70}$$

given the initial condition $\mu_{\pi,0} = 0$, as π_0 is free to jump.

The dynamics of the co-state on government debt is given by

$$\dot{\mu}_{b,t} - (\rho+\theta)\mu_{b,t} = \theta^* \Upsilon (b_t - b^n) + \mu_{\pi,t}\theta^* \kappa \Phi. \tag{71}$$

The dynamics of the co-state on the output gap is given by

$$\dot{\mu}_{x,t} - (\rho+\theta)\mu_{x,t} = \alpha x_t + \mu_{\pi,t}\kappa, \tag{72}$$

given the initial condition $\mu_{x,0} = 0$, as x_0 is free to jump.

The optimality condition for the real interest rate is

$$\mu_{b,t} + \mu_{x,t} = 0. \tag{73}$$

Real and nominal rates. Combining the optimality condition for debt and output gap, we obtain

$$\theta^* \Upsilon (b_t - b^n) + \alpha x_t = -\mu_{\pi,t}\kappa(1 + \theta^* \Phi). \tag{74}$$

Using the fact that $\mu_{\pi,0} = 0$, we obtain

$$x_0 = -\frac{\theta^* \Upsilon}{\alpha}(b_0 - b^n). \tag{75}$$

Hence, it is optimal to have a negative output gap in period 0 to compensate for the effects of the initial high value of debt, $b_0 - b^n$.

Differentiating the expression above with respect to time, we obtain

$$\dot{\mu}_{\pi,t} = -\frac{\theta^* \Upsilon + \alpha}{\kappa(1+\theta^*\Phi)}\dot{b}_t + \frac{\alpha \psi_t}{\kappa(1+\theta^*\Phi)}. \tag{76}$$

Combining the previous expression with the dynamics for $\mu_{\pi,t}$, we obtain

$$-\frac{\theta^* \Upsilon + \alpha}{\kappa(1+\theta^*\Phi)}\dot{b}_t + \frac{\alpha \psi_t}{\kappa(1+\theta^*\Phi)} = \beta \pi_t. \tag{77}$$

Rearranging the expression above, we obtain

$$\dot{b}_t = -\beta \frac{\kappa(1+\theta^*\Phi)}{\theta^* \Upsilon + \alpha}\pi_t + \frac{\alpha}{\theta^* \Upsilon + \alpha}\psi_t = r_t - \rho + \psi_t. \tag{78}$$

The real rate is given by

$$r_t - \rho = -\beta \frac{\kappa(1+\theta^*\Phi)}{\theta^* \Upsilon + \alpha}\pi_t - \frac{\theta^* \Upsilon}{\theta^* \Upsilon + \alpha}\psi_t, \tag{79}$$

and the nominal interest rate is given by

$$i_t = \rho + \left[1 - \beta \frac{\kappa(1+\theta^*\Phi)}{\theta^* \Upsilon + \alpha}\right]\pi_t - \frac{\theta^* \Upsilon}{\theta^* \Upsilon + \alpha}\psi_t. \tag{80}$$

Dynamics under the optimal policy. Using the expression for $x_t = x_0 + b_t - b_0 - \hat{\psi}_t$, we can write a dynamic system for π_t and b_t

$$\begin{bmatrix} \dot{\pi}_t \\ \dot{b}_t \end{bmatrix} = \begin{bmatrix} \rho + \theta & -\kappa(1+\theta^*\Phi) \\ -\frac{\beta\kappa(1+\theta^*\Phi)}{\theta^*\Upsilon + \alpha} & 0 \end{bmatrix} \begin{bmatrix} \pi_t \\ b_t - b^n \end{bmatrix} + \begin{bmatrix} \kappa(\hat{\psi}_t + b_0 - b^n - x_0) \\ \frac{\alpha}{\theta^*\Upsilon+\alpha}\psi_t \end{bmatrix}. \tag{81}$$

As b_0 is given and π_0 can jump, there is a unique bounded solution to the system above if the system has a positive eigenvalue and a negative eigenvalue. The eigenvalues of the system satisfy the condition

$$(\rho + \theta - \lambda)(-\lambda) - \beta \frac{\kappa^2(1+\theta^*\Phi)^2}{\theta^* \Upsilon + \alpha} = 0 \Rightarrow \lambda^2 - [\rho + \theta]\,\lambda - \beta \frac{\kappa^2(1+\theta^*\Phi)^2}{\theta^* \Upsilon + \alpha} = 0.$$

Denote the eigenvalues of the system by $\overline{\lambda} > 0$ and $\underline{\lambda} < 0$. For the case $\theta_f = \theta$, the eigenvalues are given by

$$\overline{\lambda} = \frac{\rho + \theta + \sqrt{(\rho+\theta)^2 + 4\beta\frac{\kappa^2(1+\theta^*\Phi)^2}{\theta^*\Upsilon+\alpha}}}{2}, \qquad \underline{\lambda} = \frac{\rho + \theta - \sqrt{(\rho+\theta)^2 + 4\beta\frac{\kappa^2(1+\theta^*\Phi)^2}{\theta^*\Upsilon+\alpha}}}{2}. \tag{82}$$

The matrix of eigenvectors and its inverse are given by

$$V = \begin{bmatrix} \frac{\kappa(1+\theta^*\Phi)}{\underline{\lambda}} & \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}} \\ 1 & 1 \end{bmatrix}, \qquad V^{-1} = \frac{\overline{\lambda}|\underline{\lambda}|}{(\overline{\lambda}-\underline{\lambda})\kappa(1+\theta^*\Phi)} \begin{bmatrix} -1 & \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}} \\ 1 & \frac{\kappa(1+\theta^*\Phi)}{|\underline{\lambda}|} \end{bmatrix}. \tag{83}$$

Let $Z_t = [\pi_t, b_t]'$ denote the vector of endogenous variables, A the matrix of coefficients, and U_t the vector of coefficients. We can then write the dynamic system as $\dot{Z}_t = AZ_t + U_t$. We can write the matrix of coefficients as $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix with the eigenvalues. Using the matrix eigendecomposition, we can decouple the system using the transformation: $z_t \equiv V^{-1}Z_t$ and $u_t \equiv V^{-1}U_t$. This gives us the system of decoupled differential equations:

$$\dot{z}_{1,t} = \overline{\lambda} z_{1,t} + u_{1,t}, \qquad \dot{z}_{2,t} = \underline{\lambda} z_{2,t} + u_{2,t}. \tag{84}$$

Integrating the first equation forward and the second backwards, we obtain

$$z_{1,t} = -\int_t^\infty e^{-\overline{\lambda}(s-t)} u_{1,s} ds, \qquad z_{2,t} = e^{\underline{\lambda}t} z_{2,0} + \int_0^t e^{\underline{\lambda}(t-s)} u_{2,s} ds. \tag{85}$$

Rotating the system back to its original coordinates, we obtain

$$\pi_t = \frac{\kappa(1+\theta^*\Phi)}{|\underline{\lambda}|} \int_t^\infty e^{-\overline{\lambda}(s-t)} u_{1,s} ds + \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}} \left[e^{\underline{\lambda}t} z_{2,0} + \int_0^t e^{\underline{\lambda}(t-s)} u_{2,s} ds \right], \tag{86}$$

and

$$b_t - b^n = -\int_t^\infty e^{-\overline{\lambda}(s-t)} u_{1,s} ds + e^{\underline{\lambda}t} z_{2,0} + \int_0^t e^{\underline{\lambda}(t-s)} u_{2,s} ds. \tag{87}$$

The shocks $u_{1,t}$ and $u_{2,t}$ are given by

$$u_{1,t} = \frac{|\underline{\lambda}|}{\overline{\lambda}-\underline{\lambda}} \left[\frac{\alpha}{\theta^*\Upsilon+\alpha} \psi_t - \frac{\overline{\lambda}}{(1+\theta^*\Phi)} (\hat{\psi}_t + b_0 - b^n - x_0) \right] \tag{88}$$

$$u_{2,t} = \frac{\overline{\lambda}}{\overline{\lambda}-\underline{\lambda}} \left[\frac{\alpha}{\theta^*\Upsilon+\alpha} \psi_t + \frac{|\underline{\lambda}|}{(1+\theta^*\Phi)} (\hat{\psi}_t + b_0 - b^n - x_0) \right], \tag{89}$$

where $\hat{\psi}_t = \frac{1-e^{-\theta_\psi t}}{\theta_\psi}\psi_0$ if $\theta_\psi > 0$ and $\hat{\psi}_t = \psi_0 t$ if $\theta_\psi = 0$.

The forward integral of $u_{1,t}$ is given by

$$\int_t^\infty e^{-\overline{\lambda}(s-t)} u_{1,s} ds = \frac{|\underline{\lambda}|}{\overline{\lambda}-\underline{\lambda}} \left[\left(\frac{\alpha}{\theta^*\Upsilon+\alpha} + \frac{\overline{\lambda}}{(1+\theta^*\Phi)} \frac{1}{\theta_\psi} \right) \frac{\psi_t}{\overline{\lambda}+\theta_\psi} - \frac{\frac{\psi_0}{\theta_\psi} + b_0 - b^n - x_0}{1+\theta^*\Phi} \right] \tag{90}$$

The backward integral of $u_{2,t}$ is given by

$$\int_0^t e^{\underline{\lambda}(t-s)}u_{2,s}ds = \frac{\overline{\lambda}}{\overline{\lambda}-\underline{\lambda}}\left[\left(\frac{\alpha}{\theta^*\Upsilon+\alpha} - \frac{|\underline{\lambda}|}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{e^{\underline{\lambda}t}-e^{-\theta_\psi t}}{\theta_\psi+\underline{\lambda}}\psi_0 + \frac{|\underline{\lambda}|}{(1+\theta^*\Phi)}\left(\frac{\psi_0}{\theta_\psi}+b_0-b^n-x_0\right)\frac{1-e^{\underline{\lambda}t}}{|\underline{\lambda}|}\right] \tag{91}$$

From the expression for $z_{1,0}$, we obtain

$$\begin{aligned}
\pi_0 &= \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}}\left[(b_0-b^n)+\frac{\overline{\lambda}-\underline{\lambda}}{|\underline{\lambda}|}\int_0^\infty e^{-\overline{\lambda}t}u_{1,t}dt\right] \\
&= \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}}\left[(b_0-b^n)+\left(\frac{\alpha}{\theta^*\Upsilon+\alpha}+\frac{\overline{\lambda}}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{\psi_0}{\overline{\lambda}+\theta_\psi} - \frac{\frac{\psi_0}{\theta_\psi}+b_0-b^n-x_0}{1+\theta^*\Phi}\right].
\end{aligned} \tag{92}$$

We can then write initial inflation as follows:

$$\pi_0 = \kappa\theta^*\frac{\alpha\Phi-\Upsilon}{\alpha\overline{\lambda}}\left[b_0-b^n+\frac{\alpha}{\theta^*\Upsilon+\alpha}\frac{\psi_0}{\overline{\lambda}+\theta_\psi}\right].$$

The initial value for $z_{2,t}$ is given by

$$z_{2,0} = \frac{\overline{\lambda}}{\overline{\lambda}-\underline{\lambda}}\left[\frac{|\underline{\lambda}|}{\kappa(1+\theta^*\Phi)}\pi_0+b_0-b^n\right].$$

Inflation is then given by

$$\pi_t = \frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}-\underline{\lambda}}\left[\left(\frac{\alpha}{\theta^*\Upsilon+\alpha}+\frac{\overline{\lambda}}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{\psi_t}{\overline{\lambda}+\theta_\psi} - \frac{\frac{\psi_0}{\theta_\psi}+b_0-b^n-x_0}{1+\theta^*\Phi}\right] \tag{93}$$

$$+\frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}-\underline{\lambda}}\left[e^{\underline{\lambda}t}\left[\frac{|\underline{\lambda}|}{\kappa(1+\theta^*\Phi)}\pi_0+b_0-b^n\right]+\left(\frac{\alpha}{\theta^*\Upsilon+\alpha}-\frac{|\underline{\lambda}|}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{e^{\underline{\lambda}t}-e^{-\theta_\psi t}}{\theta_\psi+\underline{\lambda}}\psi_0\right] \tag{94}$$

$$+\frac{\kappa(1+\theta^*\Phi)}{\overline{\lambda}-\underline{\lambda}}\left[\frac{1-e^{\underline{\lambda}t}}{(1+\theta^*\Phi)}\left(\frac{\psi_0}{\theta_\psi}+b_0-b^n-x_0\right)\right]. \tag{95}$$

Rearranging the expression above, we obtain

$$\pi_t = \kappa\theta^*\frac{\alpha\Phi-\Upsilon}{\theta^*\Upsilon+\alpha}\frac{e^{\underline{\lambda}t}-e^{-\theta_\psi t}}{(\overline{\lambda}+\theta_\psi)(\underline{\lambda}+\theta_\psi)}\psi_0 + \frac{\kappa\theta^*}{\overline{\lambda}}\frac{\alpha\Phi-\Upsilon}{\theta^*\Upsilon+\alpha}\left[\frac{\theta^*\Upsilon+\alpha}{\alpha}(b_0-b^n)+\frac{\psi_0}{\theta_\psi+\overline{\lambda}}\right]e^{\underline{\lambda}t}. \tag{96}$$

Government debt is given by

$$b_t-b^n = \frac{\overline{\lambda}}{\kappa(1+\theta^*\Phi)}\pi_t - \left[\left(\frac{\alpha}{\theta^*\Upsilon+\alpha}+\frac{\overline{\lambda}}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{\psi_t}{\overline{\lambda}+\theta_\psi}-\frac{\frac{\psi_0}{\theta_\psi}+b_0-b^n-x_0}{1+\theta^*\Phi}\right]. \tag{97}$$

Plugging the value for π_t, we obtain

$$b_t - b^n = b_0 - b^n - \theta^* \frac{\alpha\Phi - \Upsilon}{\alpha(1+\theta^*\Phi)}(b_0 - b^n)(1 - e^{\underline{\lambda}t}) - \left(\frac{\alpha}{\theta^*\Upsilon + \alpha} - \frac{1}{(1+\theta^*\Phi)}\right)\frac{1 - e^{-\underline{\lambda}t}}{\overline{\lambda} + \theta_\psi}\frac{\overline{\lambda}}{\underline{\lambda} + \theta_\psi}\psi_0$$
$$+ \left(\frac{\alpha}{\theta^*\Upsilon + \alpha} + \frac{\overline{\lambda}}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{1 - e^{-\theta_\psi t}}{\overline{\lambda} + \theta_\psi}\psi_0 + \left(\frac{\alpha}{\theta^*\Upsilon + \alpha} - \frac{1}{(1+\theta^*\Phi)}\right)\frac{1 - e^{-\theta_\psi t}}{\overline{\lambda} + \theta_\psi}\frac{\overline{\lambda}}{\underline{\lambda} + \theta_\psi}\psi_0 \tag{98}$$

Rearranging the expression above, we obtain

$$b_t - b^n = b_0 - b^n - \theta^* \frac{\alpha\Phi - \Upsilon}{\alpha(1+\theta^*\Phi)}\left[(b_0 - b^n) + \frac{\overline{\lambda}}{(\overline{\lambda} + \theta_\psi)(\underline{\lambda} + \theta_\psi)}\frac{\psi_0}{\theta^*\Upsilon + \alpha}\right](1 - e^{\underline{\lambda}t})$$
$$+ \left(\frac{\alpha(\overline{\lambda} + \underline{\lambda} + \theta_\psi)}{\theta^*\Upsilon + \alpha} + \frac{\overline{\lambda}\underline{\lambda}}{(1+\theta^*\Phi)}\frac{1}{\theta_\psi}\right)\frac{1 - e^{-\theta_\psi t}}{(\overline{\lambda} + \theta_\psi)(\underline{\lambda} + \theta_\psi)}\psi_0. \tag{99}$$

The long-run level of debt in Phase I is given by

$$b_t^{lr} - b^n = \frac{\alpha + \theta^*\Upsilon}{\alpha(1+\theta^*\Phi)}(b_0 - b^n) + \dots \tag{100}$$

E.1 The general case

We consider next where we allow for households' expectation effects. Moreover, we allow the planner's beliefs to differ from the beliefs of households or firms. In this case, the planner's problem is given by:

$$\max_{\{[\pi_t, b_t, x_t, r_t]_0^\infty\}} -\frac{1}{2}\int_{t_0}^{\infty} e^{-(\rho+\theta)(t-t_0)}\left[\alpha x_t^2 + \beta\pi_t^2 + \theta^*\Upsilon(b_t - b^n)^2\right]dt, \tag{101}$$

subject to

$$\dot{\pi}_t = (\rho + \theta_f)\pi_t - \kappa x_t - \theta_f^*\kappa\Phi(b_t - b^n) \tag{102}$$
$$\dot{b}_t = r_t - \rho + \psi_t \tag{103}$$
$$\dot{x}_t = r_t - \rho + \theta_h x_t - \theta_h^*(b_t - b^n), \tag{104}$$

given b_{t_0}.

Optimality conditions. The Hamiltonian to this problem is given by

$$\mathcal{H} = -\frac{1}{2}\left[\alpha x_t^2 + \beta\pi_t^2 + \theta^*\Upsilon(b_t - b^n)^2\right] + \mu_{\pi,t}\left[(\rho+\theta_f)\pi_t - \kappa x_t - \theta_f^*\kappa\Phi(b_t - b^n)\right] \tag{105}$$
$$+ \mu_{b,t}\left[r_t - \rho + \psi_t\right] + \mu_{x,t}[r_t - \rho + \theta_h x_t - \theta_h^*(b_t - b^n)], \tag{106}$$

The dynamics of the co-state on inflation is given by

$$\dot{\mu}_{\pi,t} - (\rho+\theta)\mu_{\pi,t} = \beta\pi_t - \mu_{\pi,t}(\rho+\theta_f), \tag{107}$$

given the initial condition $\mu_{\pi,0} = 0$, as π_0 is free to jump.

The dynamics of the co-state on government debt is given by

$$\dot{\mu}_{b,t} - (\rho + \theta)\mu_{b,t} = \theta^* \Upsilon(b_t - b^n) + \mu_{\pi,t}\theta_f^* \kappa \Phi + \mu_{x,t}\theta_h^*. \tag{108}$$

The dynamics of the co-state on the output gap is given by

$$\dot{\mu}_{x,t} - (\rho + \theta)\mu_{x,t} = \alpha x_t + \mu_{\pi,t}\kappa - \mu_{x,t}\theta_h, \tag{109}$$

given the initial condition $\mu_{x,0} = 0$, as x_0 is free to jump.

The optimality condition for the real interest rate is

$$\mu_{b,t} + \mu_{x,t} = 0. \tag{110}$$

Real and nominal rates. Combining the optimality condition for debt and output gap, we obtain

$$\theta^* \Upsilon(b_t - b^n) + \alpha x_t = -\mu_{\pi,t}\kappa(1 + \theta^* \Phi) + \mu_{x,t}\overline{\theta}_h. \tag{111}$$

Using the fact that $\mu_{\pi,0} = 0$ and $\mu_{x,0} = 0$, we obtain

$$x_0 = -\frac{\theta^* \Upsilon}{\alpha}(b_0 - b^n). \tag{112}$$

Hence, it is optimal to have a negative output gap in period 0 to compensate for the effects of the initial high value of debt, $b_0 - b^n$.

Differentiating the expression above with respect to time, we obtain

$$\dot{\mu}_{\pi,t} = -\frac{\theta^* \Upsilon + \alpha}{\kappa(1 + \theta^* \Phi)}\dot{b}_t + \frac{\alpha \psi_t}{\kappa(1 + \theta^* \Phi)}. \tag{113}$$

Combining the previous expression with the dynamics for $\mu_{\pi,t}$, we obtain

$$-\frac{\theta^* \Upsilon + \alpha}{\kappa(1 + \theta^* \Phi)}\dot{b}_t + \frac{\alpha \psi_t}{\kappa(1 + \theta^* \Phi)} = \beta \pi_t. \tag{114}$$

Rearranging the expression above, we obtain

$$\dot{b}_t = -\beta \frac{\kappa(1 + \theta^* \Phi)}{\theta^* \Upsilon + \alpha}\pi_t + \frac{\alpha}{\theta^* \Upsilon + \alpha}\psi_t = r_t - \rho + \psi_t. \tag{115}$$

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